VERSION 0.9 NCS

GRADE 11 MATHEMATICS

WRITTEN BY VOLUNTEERS

SIYAVULA
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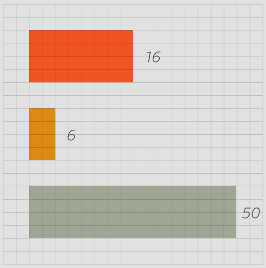

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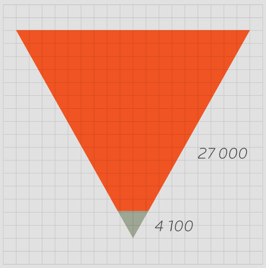

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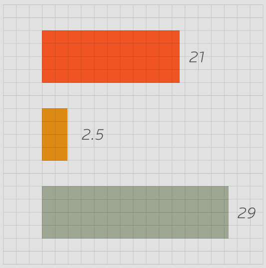

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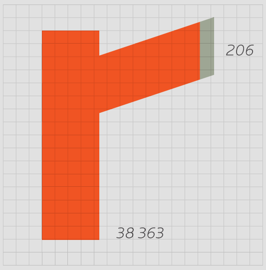

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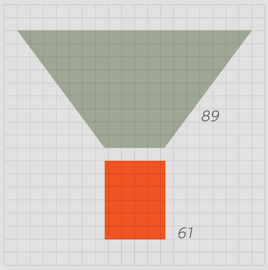

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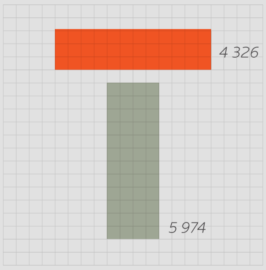

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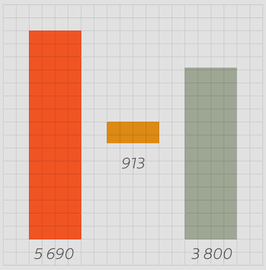

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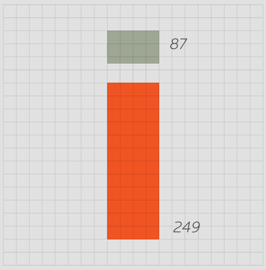

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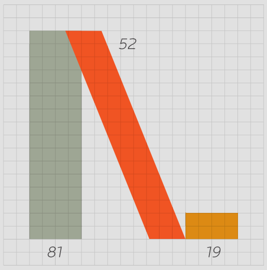

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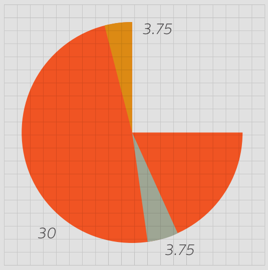

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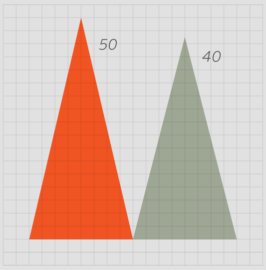

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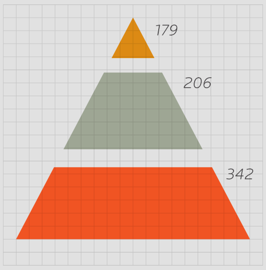

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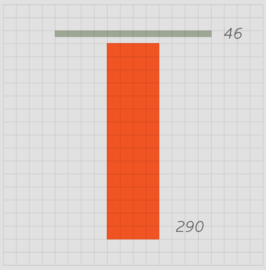

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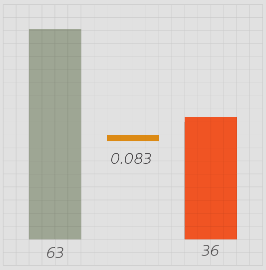

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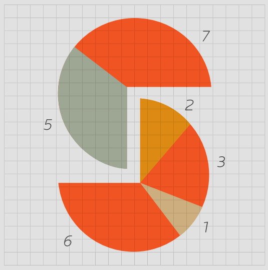

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Everything Maths

Grade 11 Mathematics

Version 0.9 – NCS

by Siyavula and volunteers

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Everything Maths

Mathematics is commonly thought of as being about numbers but mathematics is actually a language! Mathematics is the language that nature speaks to us in. As we learn to understand and speak this language, we can discover many of nature's secrets. Just as understanding someone's language is necessary to learn more about them, mathematics is required to learn about all aspects of the world – whether it is physical sciences, life sciences or even finance and economics.

The great writers and poets of the world have the ability to draw on words and put them together in ways that can tell beautiful or inspiring stories. In a similar way, one can draw on mathematics to explain and create new things. Many of the modern technologies that have enriched our lives are greatly dependent on mathematics. DVDs, Google searches, bank cards with PIN numbers are just some examples. And just as words were not created specifically to tell a story but their existence enabled stories to be told, so the mathematics used to create these technologies was not developed for its own sake, but was available to be drawn on when the time for its application was right.

There is in fact not an area of life that is not affected by mathematics. Many of the most sought after careers depend on the use of mathematics. Civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programmes useful.

But, even in our daily lives mathematics is everywhere – in our use of distance, time and money. Mathematics is even present in art, design and music as it informs proportions and musical tones. The greater our ability to understand mathematics, the greater our ability to appreciate beauty and everything in nature. Far from being just a cold and abstract discipline, mathematics embodies logic, symmetry, harmony and technological progress. More than any other language, mathematics is everywhere and universal in its application.

See introductory video by Dr. Mark Horner: ▶ VMiwd at www.everythingmaths.co.za

More than a regular textbook

Everything Maths is not just a Mathematics textbook. It has everything you expect from your regular printed school textbook, but comes with a whole lot more. For a start, you can download or read it on-line on your mobile phone, computer or iPad, which means you have the convenience of accessing it wherever you are.

We know that some things are hard to explain in words. That is why every chapter comes with video lessons and explanations which help bring the ideas and concepts to life. Summary presentations at the end of every chapter offer an overview of the content covered, with key points highlighted for easy revision.

All the exercises inside the book link to a service where you can get more practice, see the full solution or test your skills level on mobile and PC.

We are interested in what you think, wonder about or struggle with as you read through the book and attempt the exercises. That is why we made it possible for you to use your mobile phone or computer to digitally pin your question to a page and see what questions and answers other readers pinned up.

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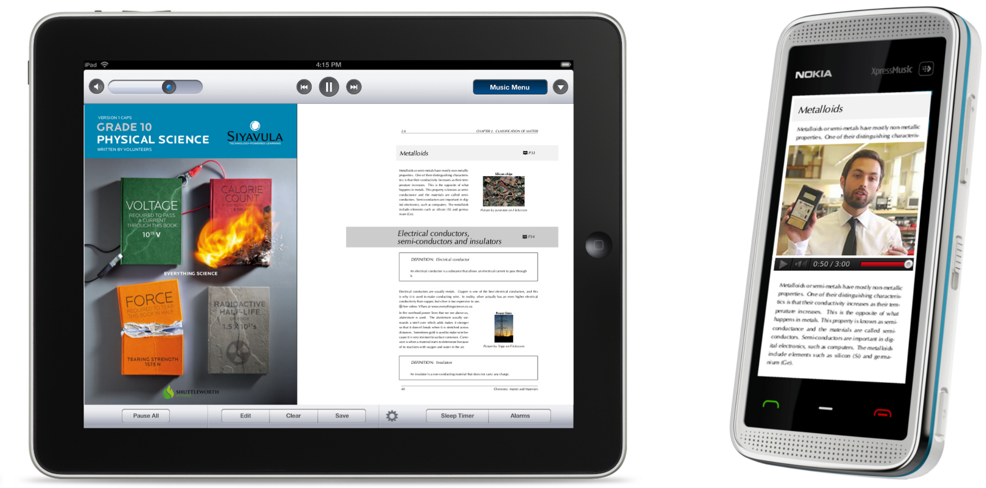


Using the icons and short-codes

Inside the book you will find these icons to help you spot where videos, presentations, practice tools and more help exist. The short-codes next to the icons allow you to navigate directly to the resources on-line without having to search for them.

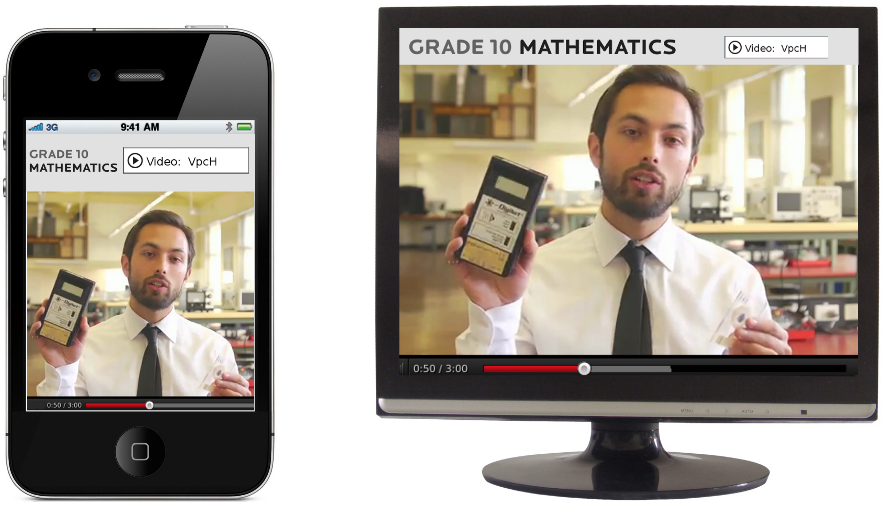
- (A123) Go directly to a section
- (V123) Video, simulation or presentation
- (P123) Practice and test your skills
- (Q123) Ask for help or find an answer

To watch the videos on-line, practise your skills or post a question, go to the *Everything Maths* website at www.everythingmaths.co.za on your mobile or PC and enter the short-code in the navigation box.

Video lessons

Look out for the video icons inside the book. These will take you to video lessons that help bring the ideas and concepts on the page to life. Get extra insight, detailed explanations and worked examples. See the concepts in action and hear real people talk about how they use maths and science in their work.

See video explanation (Video: V123)



Video exercises

Wherever there are exercises in the book you will see icons and short-codes for video solutions, practice and help. These short-codes will take you to video solutions of select exercises to show you step-by-step how to solve such problems.

See video exercise (Video: V123)

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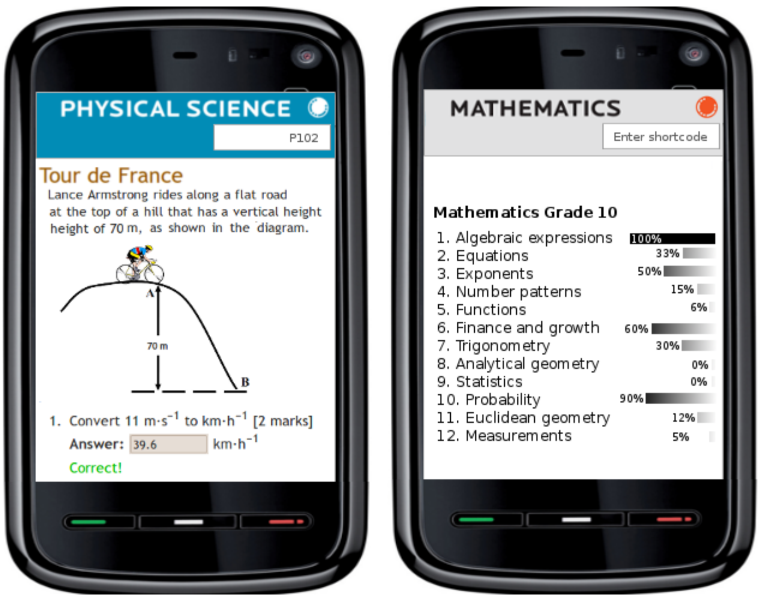
Practice and test your skills

One of the best ways to prepare for your tests and exams is to practice answering the same kind of questions you will be tested on. At every set of exercises you will see a practice icon and short-code. This on-line practice for **mobile** and **PC** will keep track of your performance and progress, give you feedback on areas which require more attention and suggest which sections or videos to look at.

See more practice (QM123)



To practice and test your skills:

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Answers to your questions

Have you ever had a question about a specific fact, formula or exercise in your textbook and wished you could just ask someone? Surely someone else in the country must have had the same question at the same place in the textbook.

Database of questions and answers

We invite you to browse our database of questions and answer for every sections and exercises in the book. Find the short-code for the section or exercise where you have a question and enter it into the short-code search box on the web or mobi-site at www.everythingmaths.co.za or www.everythingscience.co.za. You will be directed to all the questions previously asked and answered for that section or exercise.

(A123) Visit this section to post or view questions

(Q123) Questions or help with a specific question

Ask an expert

Can't find your question or the answer to it in the questions database? Then we invite you to try our service where you can send your question directly to an expert who will reply with an answer. Again, use the short-code for the section or exercise in the book to identify your problem area.

Contents

Introduction to the Book 1

1.1 The Language of Mathematics EMBA

The purpose of any language, like English or Zulu, is to make it possible for people to communicate. All languages have an alphabet, which is a group of letters that are used to make up words. There are also rules of grammar which explain how words are supposed to be used to build up sentences. This is needed because when a sentence is written, the person reading the sentence understands exactly what the writer is trying to explain. Punctuation marks (like a full stop or a comma) are used to further clarify what is written.

Mathematics is a language, specifically it is the language of Science. Like any language, mathematics has letters (known as numbers) that are used to make up words (known as expressions), and sentences (known as equations). The punctuation marks of mathematics are the different signs and symbols that are used, for example, the plus sign ($+$), the minus sign ($-$), the multiplication sign ($\times$), the equals sign ($=$) and so on. There are also rules that explain how the numbers should be used together with the signs to make up equations that express some meaning.

See introductory video: VMinh at www.everythingmaths.co.za

Exponents 2

2.1 Introduction

EMBB

In Grade 10 we studied exponential numbers and learnt that there are six laws that make working with exponential numbers easier. There is one law that we did not study in Grade 10. This will be described here.

See introductory video: VMeac at www.everythingmaths.co.za

2.2 Laws of Exponents

EMBC

In Grade 10, we worked only with indices that were integers. What happens when the index is not an integer, but is a rational number? This leads us to the final law of exponents,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \tag{2.1}$$

Exponential Law 7: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

EMBD

We say that x is an nth root of b if $x^n = b$ and we write $x = \sqrt[n]{b}$. n^{th} roots written with the radical symbol, $\sqrt{\ }$, are referred to as surds. For example, $(-1)^4 = 1$, so -1 is a 4^{th} root of 1. Using Law 6 from Grade 10, we notice that

$$(a^{\frac{m}{n}})^n = a^{\frac{m}{n} \times n} = a^m \tag{2.2}$$

therefore $a^{\frac{m}{n}}$ must be an nth root of a^m. We can therefore say

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \tag{2.3}$$

For example,

$$2^{\frac{2}{3}} = \sqrt[3]{2^2}$$

A number may not always have a real nth root. For example, if $n = 2$ and $a = -1$, then there is no real number such that $x^2 = -1$ because $x^2 \geq 0$ for all real numbers x.

Extension:	*Complex Numbers*

There are numbers which can solve problems like $x^2 = -1$, but they are beyond the scope of this book. They are called *complex numbers*.

It is also possible for more than one nth root of a number to exist. For example, $(-2)^2 = 4$ and $2^2 = 4$, so both -2 and 2 are 2^{nd} (square) roots of 4. Usually, if there is more than one root, we choose the positive real solution and move on.

Example 1: *Rational Exponents*

QUESTION

Simplify without using a calculator:

$$\left(\frac{5}{4^{-1} - 9^{-1}}\right)^{\frac{1}{2}}$$

SOLUTION

Step 1 : ***Rewrite negative exponents as numbers with positive indices***

$$= \left(\frac{5}{\frac{1}{4} - \frac{1}{9}}\right)^{\frac{1}{2}}$$

Step 2 : ***Simplify inside brackets***

$$\begin{aligned} &= \left(\frac{5}{\frac{9-4}{36}}\right)^{\frac{1}{2}} \\ &= \left(\frac{5}{1} \div \frac{5}{36}\right)^{\frac{1}{2}} \\ &= (6^2)^{\frac{1}{2}} \end{aligned}$$

Step 3 : ***Apply exponential Law 6***

$$= 6$$

Example 2: *More rational Exponents*

QUESTION

Simplify:

$$(16x^4)^{\frac{3}{4}}$$

SOLUTION

Step 1 : ***Convert the number coefficient to a product of it's prime factors***

$$= (2^4x^4)^{\frac{3}{4}}$$

Step 2 : ***Apply exponential laws***

$$\begin{aligned} &= 2^{4\times\frac{3}{4}}.x^{4\times\frac{3}{4}} \\ &= 2^3.x^3 \\ &= 8x^3 \end{aligned}$$

See video: VMebb at www.everythingmaths.co.za

Exercise 2 - 1

Use all the laws to:

1. Simplify:

   (a) $(x^0) + 5x^0 - (0{,}25)^{-0{,}5} + 8^{\frac{2}{3}}$

   (b) $s^{\frac{1}{2}} \div s^{\frac{1}{3}}$

   (c) $(64m^6)^{\frac{2}{3}}$

   (d) $\dfrac{12m^{\frac{7}{9}}}{8m^{-\frac{11}{9}}}$

2. Re-write the following expression as a power of x:

$$x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}$$

More practice video solutions or help at www.everythingmaths.co.za

(1.) 016e (2.) 016f

2.3 Exponentials in the Real World

EMBE

In Grade 10 Finance, you used exponentials to calculate different types of interest, for example on a savings account or on a loan and compound growth.

Example 3: Exponentials in the Real world

QUESTION

A type of bacteria has a very high exponential growth rate at 80% every hour. If there are 10 bacteria, determine how many there will be in five hours, in one day and in one week?

SOLUTION

Step 1 : ***Population = Initial population*** $\times (1 +$ ***growth percentage***$)^{time\ period\ in\ hours}$
Therefore, in this case:
$Population = 10(1{,}8)^{n}$, where n = number of hours

Step 2 : ***In 5 hours***
$Population = 10(1{,}8)^{5} = 189$

Step 3 : ***In 1 day = 24 hours***
$Population = 10(1{,}8)^{24} = 13\ 382\ 588$

Step 4 : ***in 1 week = 168 hours***
$Population = 10(1{,}8)^{168} = 7{,}687 \times 10^{43}$
Note this answer is given in scientific notation as it is a very big number.

Example 4: *More Exponentials in the Real world*

QUESTION

A species of extremely rare, deep water fish has an very long lifespan and rarely has children. If there are a total 821 of this type of fish and their growth rate is 2% each month, how many will there be in half of a year? What will the population be in ten years and in one hundred years?

SOLUTION

Step 1 : ***Population = Initial population*** $\times (1+$ ***growth percentage***$)^{time\ period\ in\ months}$
Therefore, in this case:
Population $= 821(1{,}02)^n$, where n = number of months

Step 2 : ***In half a year = 6 months***
Population $= 821(1{,}02)^6 = 925$

Step 3 : ***In 10 years = 120 months***
Population $= 821(1{,}02)^{120} = 8\ 838$

Step 4 : ***in 100 years = 1 200 months***
Population $= 821(1{,}02)^{1\ 200} = 1{,}716 \times 10^{13}$
Note this answer is also given in scientific notation as it is a very big number.

Chapter 2 End of Chapter Exercises

1. Simplify as far as possible:
   (a) $8^{-\frac{2}{3}}$
   (b) $\sqrt{16} + 8^{-\frac{2}{3}}$
2. Simplify:

   a. $(x^3)^{\frac{4}{3}}$
   b. $(s^2)^{\frac{1}{2}}$
   c. $(m^5)^{\frac{5}{3}}$
   d. $(-m^2)^{\frac{4}{3}}$
   e. $-(m^2)^{\frac{4}{3}}$
   f. $(3y^{\frac{4}{3}})^4$

3. Simplify as much as you can:

$$\frac{3a^{-2}b^{15}c^{-5}}{(a^{-4}b^3c)^{\frac{-5}{2}}}$$

4. Simplify as much as you can:
$$\left(9a^6b^4\right)^{\frac{1}{2}}$$

5. Simplify as much as you can:
$$\left(a^{\frac{3}{2}}b^{\frac{3}{4}}\right)^{16}$$

6. Simplify:
$$x^3\sqrt{x}$$

7. Simplify:
$$\sqrt[3]{x^4b^5}$$

8. Re-write the following expression as a power of x:
$$\frac{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}}{\sqrt[3]{x}}$$

More practice video solutions or help at www.everythingmaths.co.za

(1.) 016g (2.) 016h (3.) 016i (4.) 016j (5.) 016k (6.) 016m
(7.) 016n (8.) 016p

Surds 3

3.1 Introduction EMBF

In the previous chapter on exponents, we saw that rational exponents are directly related to surds. We will discuss surds and the laws that govern them further here. While working with surds, always remember that they are directly related to exponents and that you can use your knowledge of one to help with understanding the other.

See introductory video: VMebn at www.everythingmaths.co.za

3.2 Surd Calculations EMBG

There are several laws that make working with surds (or roots) easier. We will list them all and then explain where each rule comes from in detail.

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab} \quad (3.1)$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad (3.2)$$

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} \quad (3.3)$$

Surd Law 1: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ EMBH

It is often useful to look at a surd in exponential notation as it allows us to use the exponential laws we learnt in Grade 10. In exponential notation, $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $\sqrt[n]{b} = b^{\frac{1}{n}}$. Then,

$$\begin{aligned} \sqrt[n]{a}\sqrt[n]{b} &= a^{\frac{1}{n}}b^{\frac{1}{n}} \quad (3.4) \\ &= (ab)^{\frac{1}{n}} \\ &= \sqrt[n]{ab} \end{aligned}$$

Some examples using this law:

1. $\sqrt[3]{16} \times \sqrt[3]{4}$
   $= \sqrt[3]{64}$
   $= 4$
2. $\sqrt{2} \times \sqrt{32}$
   $= \sqrt{64}$
   $= 8$

3. $\sqrt{a^2b^3} \times \sqrt{b^5c^4}$
   $= \sqrt{a^2b^8c^4}$
   $= ab^4c^2$

Surd Law 2: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ EMBI

If we look at $\sqrt[n]{\frac{a}{b}}$ in exponential notation and apply the exponential laws then,

$$\begin{aligned} \sqrt[n]{\frac{a}{b}} &= \left(\frac{a}{b}\right)^{\frac{1}{n}} \\ &= \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \\ &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \end{aligned} \tag{3.5}$$

Some examples using this law:

1. $\sqrt{12} \div \sqrt{3}$
   $= \sqrt{4}$
   $= 2$

2. $\sqrt[3]{24} \div \sqrt[3]{3}$
   $= \sqrt[3]{8}$
   $= 2$

3. $\sqrt{a^2b^{13}} \div \sqrt{b^5}$
   $= \sqrt{a^2b^8}$
   $= ab^4$

Surd Law 3: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ EMBJ

If we look at $\sqrt[n]{a^m}$ in exponential notation and apply the exponential laws then,

$$\begin{aligned} \sqrt[n]{a^m} &= (a^m)^{\frac{1}{n}} \\ &= a^{\frac{m}{n}} \end{aligned} \tag{3.6}$$

For example,

$$\begin{aligned} \sqrt[6]{2^3} &= 2^{\frac{3}{6}} \\ &= 2^{\frac{1}{2}} \\ &= \sqrt{2} \end{aligned}$$

Like and Unlike Surds

EMBK

Two surds $\sqrt[m]{a}$ and $\sqrt[n]{b}$ are called *like surds* if $m = n$, otherwise they are called *unlike surds*. For example $\sqrt{2}$ and $\sqrt{3}$ are like surds, however $\sqrt{2}$ and $\sqrt[3]{2}$ are unlike surds. An important thing to realise about the surd laws we have just learnt is that the surds in the laws are all like surds.

If we wish to use the surd laws on unlike surds, then we must first convert them into like surds. In order to do this we use the formula

$$\sqrt[n]{a^m} = \sqrt[bn]{a^{bm}} \tag{3.7}$$

to rewrite the unlike surds so that bn is the same for all the surds.

Example 1: *Like and Unlike Surds*

QUESTION

Simplify to like surds as far as possible, showing all steps: $\sqrt[3]{3} \times \sqrt[5]{5}$

SOLUTION

Step 1 : ***Find the common root***

$$= \sqrt[15]{3^5} \times \sqrt[15]{5^3}$$

Step 2 : ***Use surd Law 1***

$$\begin{aligned} &= \sqrt[15]{3^5.5^3} \\ &= \sqrt[15]{243 \times 125} \\ &= \sqrt[15]{30375} \end{aligned}$$

Simplest Surd Form

EMBL

In most cases, when working with surds, answers are given in simplest surd form. For example,

$$\begin{aligned}\sqrt{50} &= \sqrt{25 \times 2}\\ &= \sqrt{25} \times \sqrt{2}\\ &= 5\sqrt{2}\end{aligned}$$

$5\sqrt{2}$ is the simplest surd form of $\sqrt{50}$.

Example 2: *Simplest surd form*

QUESTION

Rewrite $\sqrt{18}$ *in the simplest surd form:*

SOLUTION

Step 1 : ***Convert the number*** 18 ***into a product of it's prime factors***

$$\begin{aligned}\sqrt{18} &= \sqrt{2 \times 9}\\ &= \sqrt{2} \times \sqrt{3^2}\end{aligned}$$

Step 2 : ***Square root all squared numbers:***

$$= 3\sqrt{2}$$

Example 3: *Simplest surd form*

QUESTION

Simplify: $\sqrt{147} + \sqrt{108}$

SOLUTION

Step 1 : ***Simplify each square root by converting each number to a product of it's prime factors***

$$\begin{aligned}\sqrt{147} + \sqrt{108} &= \sqrt{49 \times 3} + \sqrt{36 \times 3} \\ &= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3}\end{aligned}$$

Step 2 : ***Square root all squared numbers***

$$= 7\sqrt{3} + 6\sqrt{3}$$

Step 3 : ***The exact same surds can be treated as "like terms" and may be added***

$$= 13\sqrt{3}$$

See video: VMecu at www.everythingmaths.co.za

Rationalising Denominators

EMBM

It is useful to work with fractions, which have rational denominators instead of surd denominators. It is possible to rewrite any fraction, which has a surd in the denominator as a fraction which has a rational denominator. We will now see how this can be achieved.

Any expression of the form $\sqrt{a}+\sqrt{b}$ (where a and b are rational) can be changed into a rational number by multiplying by $\sqrt{a} - \sqrt{b}$ (similarly $\sqrt{a} - \sqrt{b}$ can be rationalised by multiplying by $\sqrt{a} + \sqrt{b}$). This is because

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b \tag{3.8}$$

which is rational (since a and b are rational).

If we have a fraction which has a denominator which looks like $\sqrt{a} + \sqrt{b}$, then we can simply multiply the fraction by $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$ to achieve a rational denominator. (Remember that $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = 1$)

$$\begin{aligned}\frac{c}{\sqrt{a} + \sqrt{b}} &= \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} \times \frac{c}{\sqrt{a} + \sqrt{b}} \\ &= \frac{c\sqrt{a} - c\sqrt{b}}{a - b}\end{aligned} \tag{3.9}$$

or similarly

$$\frac{c}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} \times \frac{c}{\sqrt{a}-\sqrt{b}} \qquad (3.10)$$
$$= \frac{c\sqrt{a}+c\sqrt{b}}{a-b}$$

Example 4: *Rationalising the Denominator*

QUESTION

Rationalise the denominator of: $\frac{5x-16}{\sqrt{x}}$

SOLUTION

Step 1 : ***Rationalise the denominator***
To get rid of $\sqrt{x}$ in the denominator, you can multiply it out by another $\sqrt{x}$. This *rationalises* the surd in the denominator. Note that $\frac{\sqrt{x}}{\sqrt{x}} = 1$, thus the equation becomes rationalised by multiplying by 1 (although its' value stays the same).

$$\frac{5x-16}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}}$$

Step 2 : ***Multiply out the numerators and denominators***
The surd is expressed in the numerator which is the preferred way to write expressions. (That's why denominators get rationalised.)

$$\frac{5x\sqrt{x}-16\sqrt{x}}{x}$$
$$= \frac{(\sqrt{x})(5x-16)}{x}$$

Example 5: *Rationalising the Denominator*

QUESTION

Rationalise the following: $\frac{5x-16}{\sqrt{y}-10}$

SOLUTION

Step 1 : ***Rationalise the denominator***

$$\frac{5x-16}{\sqrt{y}-10} \times \frac{\sqrt{y}+10}{\sqrt{y}+10}$$

Step 2 : ***Multiply out the numerators and denominators***

$$\frac{5x\sqrt{y}-16\sqrt{y}+50x-160}{y-100}$$

All the terms in the numerator are different and cannot be simplified and the denominator does not have any surds in it anymore.

Example 6: *Rationalise the denominator*

QUESTION

Simplify the following: $\frac{y-25}{\sqrt{y}+5}$

SOLUTION

Step 1 : ***Rationalise the denominator***

$$\frac{y-25}{\sqrt{y}+5} \times \frac{\sqrt{y}-5}{\sqrt{y}-5}$$

Step 2 : ***Multiply out the numerators and denominators***

$$\begin{aligned} \frac{y\sqrt{y}-25\sqrt{y}-5y+125}{y-25} &= \frac{\sqrt{y}(y-25)-5(y-25)}{(y-25)} \\ &= \frac{(y-25)(\sqrt{y}-25)}{(y-25)} \\ &= \sqrt{y}-25 \end{aligned}$$

See video: VMeea at www.everythingmaths.co.za

Chapter 3 End of Chapter Exercises

1. Expand:
$$(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})$$
2. Rationalise the denominator:
$$\frac{10}{\sqrt{x}-\frac{1}{x}}$$
3. Write as a single fraction:
$$\frac{3}{2\sqrt{x}}+\sqrt{x}$$
4. Write in simplest surd form:
   (a) $\sqrt{72}$
   (b) $\sqrt{45}+\sqrt{80}$
   (c) $\frac{\sqrt{48}}{\sqrt{12}}$
   (d) $\frac{\sqrt{18}\div\sqrt{72}}{\sqrt{8}}$
   (e) $\frac{4}{(\sqrt{8}\div\sqrt{2})}$
   (f) $\frac{16}{(\sqrt{20}\div\sqrt{12})}$
5. Expand and simplify:
$$(2+\sqrt{2})^2$$
6. Expand and simplify:
$$(2+\sqrt{2})(1+\sqrt{8})$$
7. Expand and simplify:
$$(1+\sqrt{3})(1+\sqrt{8}+\sqrt{3})$$
8. Simplify, without use of a calculator:
$$\sqrt{5}(\sqrt{45}+2\sqrt{80})$$
9. Simplify:
$$\sqrt{98x^6}+\sqrt{128x^6}$$
10. Write the following with a rational denominator:
$$\frac{\sqrt{5}+2}{\sqrt{5}}$$
11. Simplify, without use of a calculator:
$$\frac{\sqrt{98}-\sqrt{8}}{\sqrt{50}}$$
12. Rationalise the denominator:
$$\frac{y-4}{\sqrt{y}-2}$$
13. Rationalise the denominator:
$$\frac{2x-20}{\sqrt{y}-\sqrt{10}}$$
14. Evaluate without using a calculator: $\left(2-\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}\times\left(2+\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}$
15. Prove (without the use of a calculator) that:
$$\sqrt{\frac{8}{3}}+5\sqrt{\frac{5}{3}}-\sqrt{\frac{1}{6}}=\frac{10\sqrt{15}+3\sqrt{6}}{6}$$

16. The use of a calculator is not permissible in this question. Simplify completely by showing all your steps: $3^{-\frac{1}{2}}\left[\sqrt{12}+\sqrt[3]{(3\sqrt{3})}\right]$

17. Fill in the blank surd-form number on the right hand side of the equation which will make the following a true statement: $-3\sqrt{6}\times -2\sqrt{24} = -\sqrt{18}\times \ldots\ldots$

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(1.) 016q	(2.) 016r	(3.) 016s	(4.) 016t	(5.) 016u	(6.) 016v
(7.) 016w	(8.) 016x	(9.) 016y	(10.) 016z	(11.) 0170	(12.) 0171
(13.) 0172	(14.) 0173	(15.) 0174	(16.) 0175	(17.) 0176	

Error Margins 4

4.1 Introduction EMBN

When rounding off, we throw away some of the digits of a number. This means that we are making an error. In this chapter we discuss how errors can grow larger than expected if we are not careful with algebraic calculations.

See introductory video: VMefg at www.everythingmaths.co.za

4.2 Rounding Off EMBO

We have seen that numbers are either rational or irrational and we have see how to round off numbers. However, in a calculation that has many steps, it is best to leave the rounding off right until the end.

For example, if you were asked to write

$$3\sqrt{3}+\sqrt{12}$$

as a decimal number correct to two decimal places, there are two ways of doing this as described in Table 4.1.

Table 4.1: Two methods of writing $3\sqrt{3}+\sqrt{12}$ as a decimal number.

☺ Method 1	☹ Method 2
$3\sqrt{3}+\sqrt{12} = 3\sqrt{3}+\sqrt{4\,.\,3}$	$3\sqrt{3}+\sqrt{12} = 3\times 1{,}73+3{,}46$
$= 3\sqrt{3}+2\sqrt{3}$	$= 5{,}19+3{,}46$
$= 5\sqrt{3}$	$= 8{,}65$
$= 5\times 1{,}732050808\ldots$	
$= 8{,}660254038\ldots$	
$= 8{,}66$	

Tip

It is best to simplify all expressions as much as possible before rounding off answers. This maintains the accuracy of your answer.

In the example we see that Method 1 gives $8{,}66$ as an answer while Method 2 gives $8{,}65$ as an answer. The answer of Method 1 is more accurate because the expression was simplified as much as possible before the answer was rounded-off.

In general, it is best to simplify any expression as much as possible, before using your calculator to work out the answer in decimal notation.

Example 1: *Simplification and Accuracy*

QUESTION

Calculate $\sqrt[3]{54}+\sqrt[3]{16}$. *Write the answer to three decimal places.*

SOLUTION

Step 1 : ***Simplify the expression***

$$\begin{aligned}\sqrt[3]{54}+\sqrt[3]{16} &= \sqrt[3]{27\,.\,2}+\sqrt[3]{8\,.\,2}\\ &= \sqrt[3]{27}\,.\,\sqrt[3]{2}+\sqrt[3]{8}\,.\,\sqrt[3]{2}\\ &= 3\sqrt[3]{2}+2\sqrt[3]{2}\\ &= 5\sqrt[3]{2}\end{aligned}$$

Step 2 : ***Convert any irrational numbers to decimal numbers***

$$5\sqrt[3]{2} = 6{,}299605249\ldots$$

Step 3 : ***Write the final answer to the required number of decimal places.***

$6{,}299605249\ldots = 6{,}300$ (to three decimal places)

$\therefore$ $\sqrt[3]{54}+\sqrt[3]{16} = 6{,}300$ (to three decimal places).

Example 2: *Simplification and Accuracy 2*

QUESTION

Calculate $\sqrt{x+1}+\frac{1}{3}\sqrt{(2x+2)-(x+1)}$ *if* $x=3{,}6$. *Write the answer to two decimal places.*

SOLUTION

Step 1 : ***Simplify the expression***

$$\begin{aligned}\sqrt{x+1}+\frac{1}{3}\sqrt{(2x+2)-(x+1)} &= \sqrt{x+1}+\frac{1}{3}\sqrt{2x+2-x-1}\\ &= \sqrt{x+1}+\frac{1}{3}\sqrt{x+1}\\ &= \frac{4}{3}\sqrt{x+1}\end{aligned}$$

Step 2 : ***Substitute the value of x into the simplified expression***

$$\begin{aligned}\frac{4}{3}\sqrt{x+1} &= \frac{4}{3}\sqrt{3{,}6+1}\\ &= \frac{4}{3}\sqrt{4{,}6}\\ &= 2{,}859681412\ldots\end{aligned}$$

Step 3 : ***Write the final answer to the required number of decimal places.***

$2{,}859681412\ldots = 2{,}86$ (To two decimal places)

$\therefore \sqrt{x+1}+\frac{1}{3}\sqrt{(2x+2)-(x+1)} = 2{,}86$ (to two decimal places) if $x = 3{,}6$.

Extension:	*Significant Figures*

In a number, each non-zero digit is a significant figure. Zeroes are only counted if they are between two non-zero digits or are at the end of the decimal part. For example, the number 2000 has one significant figure (the 2), but 2000,0 has five significant figures. Estimating a number works by removing significant figures from your number (starting from the right) until you have the desired number of significant figures, rounding as you go. For example 6,827 has four significant figures, but if you wish to write it to three significant figures it would mean removing the 7 and rounding up, so it would be 6,83. It is important to know when to estimate a number and when not to. It is usually good practise to only estimate numbers when it is absolutely necessary, and to instead use symbols to represent certain irrational numbers (such as π); approximating them only at the very end of a calculation. If it is necessary to approximate a number in the middle of a calculation, then it is often good enough to approximate to a few decimal places.

Chapter 4 End of Chapter Exercises

1. Calculate:
   (a) $\sqrt{16}\sqrt{72}$ to three decimal places
   (b) $\sqrt{25}+\sqrt{2}$ to one decimal place
   (c) $\sqrt{48}\sqrt{3}$ to two decimal places
   (d) $\sqrt{64}+\sqrt{18}\sqrt{12}$ to two decimal places

(e) $\sqrt{4} + \sqrt{20}\sqrt{18}$ to six decimal places
(f) $\sqrt{3} + \sqrt{5}\sqrt{6}$ to one decimal place

2. Calculate:
   (a) $\sqrt{x2}$, if $x = 3{,}3$. Write the answer to four decimal places.
   (b) $\sqrt{4 + x}$, if $x = 1{,}423$. Write the answer to two decimal places.
   (c) $\sqrt{x + 3} + \sqrt{x}$, if $x = 5{,}7$. Write the answer to eight decimal places.
   (d) $\sqrt{2x5} + \frac{1}{2}\sqrt{x + 1}$, if $x = 4{,}91$. Write the answer to five decimal places.
   (e) $\sqrt{3x1 + (4x + 3)}\sqrt{x + 5}$, if $x = 3{,}6$. Write the answer to six decimal places.
   (f) $\sqrt{2x + 5(x1) + (5x + 2)} + \frac{1}{4}\sqrt{4 + x}$, if $x = 1{,}09$. Write the answer to one decimal place

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(1.) 02sf (2.) 02sg

Quadratic Sequences 5

5.1 Introduction EMBP

In Grade 10 you learned about arithmetic sequences, where the difference between consecutive terms is constant. In this chapter we learn about quadratic sequences, where the difference between consecutive terms is not constant, but follows its own pattern.

See introductory video: VMeka at www.everythingmaths.co.za

5.2 What is a Quadratic Sequence? EMBQ

> *DEFINITION: Quadratic Sequence*
>
> A quadratic sequence is a sequence of numbers in which the second difference between each consecutive term is constant. This called a common second difference.

For example,

$$1;\ 2;\ 4;\ 7;\ 11;\ \ldots \tag{5.1}$$

is a quadratic sequence. Let us see why.

The first difference is calculated by finding the difference between consecutive terms:

```
1       2       4       7       11
  \   /   \   /   \   /   \   /
   +1      +2      +3      +4
```

We then work out the *second differences*, which are simply obtained by taking the difference between the consecutive differences $\{1;\ 2;\ 3;\ 4;\ \ldots\}$ obtained above:

```
1       2       3       4
  \   /   \   /   \   /
   +1      +1      +1
```

We then see that the second differences are equal to 1. Thus, Equation (5.1) is a *quadratic sequence*.

Note that the differences between consecutive terms (that is, the first differences) of a quadratic sequence form a sequence where there is a constant difference between consecutive terms. In the above example, the sequence of $\{1;\ 2;\ 3;\ 4;\ \ldots\}$, which is formed by taking the differences between consecutive terms of Equation (5.1), has a linear formula of the kind $ax + b$.

Exercise 5 - 1

The following are examples of quadratic sequences:

1. $3;\ 6;\ 10;\ 15;\ 21;\ \ldots$

2. $4;\ 9;\ 16;\ 25;\ 36;\ \ldots$

3. $7;\ 17;\ 31;\ 49;\ 71;\ \ldots$

4. $2;\ 10;\ 26;\ 50;\ 82;\ \ldots$

5. $31;\ 30;\ 27;\ 22;\ 15;\ \ldots$

Calculate the common second difference for each of the above examples.

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(1.-5.) 01zm

General Case

If the sequence is quadratic, the n^{th} term should be $T_n = an^2 + bn + c$

$TERMS$	$a+b+c$		$4a+2b+c$		$9a+3b+c$		$16a+4b+c$
1st difference		$3a+b$		$5a+b$		$7a+b$	
2nd difference			$2a$		$2a$		

In each case, the second difference is $2a$. This fact can be used to find a, then b then c.

Example 1: *Quadratic sequence*

QUESTION

Write down the next two terms and find a formula for the n^{th} *term of the sequence* $5; 12; 23; 38; \ldots$

SOLUTION

Step 1 : ***Find the first differences between the terms***

5		12		23		38
	+7		+11		+15	

i.e. $7; 11; 15$.

Step 2 : ***Find the second differences between the terms***

7 11 15

+4 +4

So the second difference is 4.
Continuing the sequence, the differences between each term will be:

...15 19 23...

+4 +4

Step 3 : ***Finding the next two terms***
The next two terms in the sequence will be:

...38 57 80...

+19 +23

So the sequence will be: $5; 12; 23; 38; 57; 80$.

Step 4 : ***Determine values for*** a, b ***and*** c

$$\begin{aligned} 2a &= 4 \\ \text{which gives} \quad a &= 2 \\ \text{And} \quad 3a + b &= 7 \\ \therefore \quad 3(2) + b &= 7 \\ b &= 7 - 6 \\ b &= 1 \\ \text{And} \quad a + b + c &= 5 \\ \therefore \quad (2) + (1) + c &= 5 \\ c &= 5 - 3 \\ c &= 2 \end{aligned}$$

Step 5 : ***Find the rule by substitution***

$$\begin{aligned} T_n &= ax^2 + bx + c \\ \therefore \quad T_n &= 2n^2 + n + 2 \end{aligned}$$

Example 2: *Quadratic Sequence*

QUESTION

The following sequence is quadratic: $8; 22; 42; 68; \ldots$ *Find the rule.*

SOLUTION

Step 1 : ***Assume that the rule is*** $an^2 + bn + c$

$TERMS$	8		22		42		68
1st difference		+14		+20		+26	
2nd difference			+6		+6		

Step 2 : ***Determine values for*** a, b ***and*** c

$$\begin{aligned} 2a &= 6 \\ \text{which gives} \quad a &= 3 \\ \text{And} \quad 3a + b &= 14 \\ \therefore \quad 9 + b &= 14 \\ b &= 5 \\ \text{And} \quad a + b + c &= 8 \\ \therefore \quad 3 + 5 + c &= 8 \\ c &= 0 \end{aligned}$$

Step 3 : ***Find the rule by substitution***

$$\begin{aligned} T_n &= ax^2 + bx + c \\ \therefore \quad T_n &= 3n^2 + 5n \end{aligned}$$

Step 4 : ***Check answer***
For

$$\begin{aligned} n = 1, \ T_1 &= 3(1)^2 + 5(1) = 8 \\ n = 2, \ T_2 &= 3(2)^2 + 5(2) = 22 \\ n = 3, \ T_3 &= 3(3)^2 + 5(3) = 42 \end{aligned}$$

Extension:	*Derivation of the* n^{th}*-term of a Quadratic Sequence*

Let the n^{th}-term for a quadratic sequence be given by

$$T_n = an^2 + bn + c \tag{5.2}$$

where a, b and c are some constants to be determined.

$$\begin{aligned} T_n &= an^2 + bn + c \\ T_1 &= a(1)^2 + b(1) + c \\ &= a + b + c \end{aligned} \tag{5.3}$$

$$\begin{aligned} T_2 &= a(2)^2 + b(2) + c \\ &= 4a + 2b + c \end{aligned} \tag{5.4}$$

$$\begin{aligned} T_3 &= a(3)^2 + b(3) + c \\ &= 9a + 3b + c \end{aligned} \tag{5.5}$$

The first difference (d) is obtained from

$$\begin{aligned} \text{Let } d &\equiv T_2 - T_1 \\ \therefore d &= 3a + b \end{aligned}$$

$$\Rightarrow b = d - 3a \tag{5.6}$$

The common second difference (D) is obtained from

$$\begin{aligned} D &= (T_3 - T_2) - (T_2 - T_1) \\ &= (5a + b) - (3a + b) \\ &= 2a \end{aligned}$$

$$\Rightarrow a = \frac{D}{2} \tag{5.7}$$

Therefore, from (5.6),

$$b = d - \frac{3}{2} \,.\, D \tag{5.8}$$

From (5.3),

$$c = T_1 - (a + b) = T_1 - \frac{D}{2} - d + \frac{3}{2} \,.\, D$$

$$\therefore c = T_1 + D - d \tag{5.9}$$

Finally, the general equation for the n^{th}-term of a quadratic sequence is given by

$$T_n = \frac{D}{2} \,.\, n^2 + (d - \frac{3}{2}\, D) \,.\, n + (T_1 - d + D) \tag{5.10}$$

Example 3: *Using a set of equations*

QUESTION

Study the following pattern: $1; 7; 19; 37; 61; \ldots$

1. *What is the next number in the sequence?*
2. *Use variables to write an algebraic statement to generalise the pattern.*
3. *What will the* 100^{th} *term of the sequence be?*

SOLUTION

Step 1 : ***The next number in the sequence***

The numbers go up in multiples of 6
$1 + 6(1) = 7$, then $7 + 6(2) = 19$
$19 + 6(3) = 37$, then $37 + 6(4) = 61$
Therefore $61 + 6(5) = 91$
The next number in the sequence is 91.

Step 2 : ***Generalising the pattern***

$TERMS$	1		7		19		37		61
1st difference		$+6$		$+12$		$+18$		$+24$	
2nd difference			$+6$		$+6$		$+6$		

The pattern will yield a quadratic equation since the second difference is constant
Therefore $T_n = an^2 + bn + c$
For the first term: $n = 1$, then $T_1 = 1$
For the second term: $n = 2$, then $T_2 = 7$
For the third term: $n = 3$, then $T_3 = 19$
etc.

Step 3 : ***Setting up sets of equations***

$$\begin{aligned} a + b + c &= 1 & \text{...eqn}(1) \\ 4a + 2b + c &= 7 & \text{...eqn}(2) \\ 9a + 3b + c &= 19 & \text{...eqn}(3) \end{aligned}$$

Step 4 : ***Solve the sets of equations***

$$\begin{aligned} \text{eqn}(2) - \text{eqn}(1): &\quad 3a + b = 6 & \text{...eqn}(4) \\ \text{eqn}(3) - \text{eqn}(2): &\quad 5a + b = 12 & \text{...eqn}(5) \\ \text{eqn}(5) - \text{eqn}(4): &\quad 2a = 6 \\ \therefore &\quad a = 3, b = -3 \ and \ c = 1 \end{aligned}$$

Step 5 : ***Final answer***

The general formula for the pattern is $T_n = 3n^2 - 3n + 1$

Step 6 : ***Term 100***

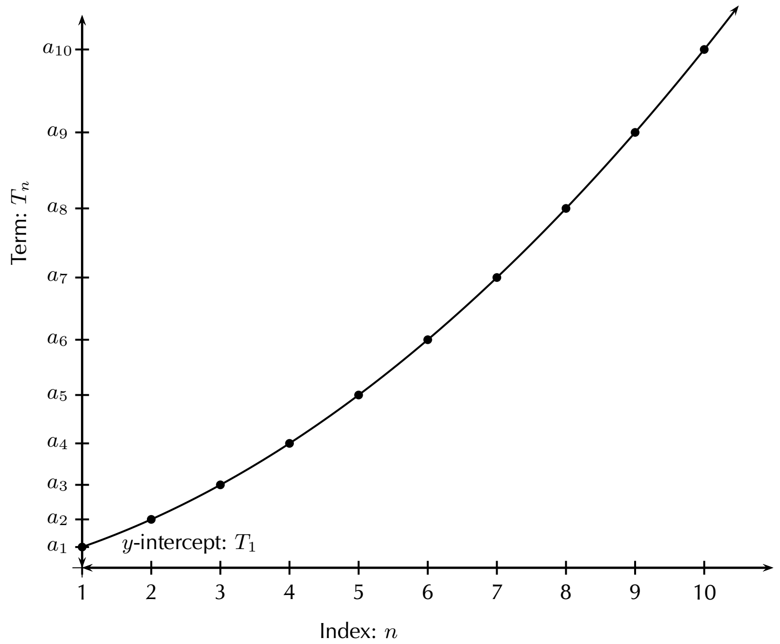
Substitute n with 100:
$3(100)^2 - 3(100) + 1 = 29\ 701$
The value for term 100 is 29 701.

Extension:	*Plotting a graph of terms of a quadratic sequence*

Plotting T_n vs. n for a quadratic sequence yields a parabolic graph.
Given the quadratic sequence,

$$3;\ 6;\ 10;\ 15;\ 21;\ \ldots$$

If we plot each of the terms vs. the corresponding index, we obtain a graph of a parabola.



Chapter 5 End of Chapter Exercises

1. Find the first five terms of the quadratic sequence defined by:

$$a_n = n^2 + 2n + 1$$

2. Determine which of the following sequences is a quadratic sequence by calculating the common second difference:
   (a) $6; 9; 14; 21; 30; \ldots$
   (b) $1; 7; 17; 31; 49; \ldots$
   (c) $8; 17; 32; 53; 80; \ldots$
   (d) $9; 26; 51; 84; 125; \ldots$
   (e) $2; 20; 50; 92; 146; \ldots$
   (f) $5; 19; 41; 71; 109; \ldots$
   (g) $2; 6; 10; 14; 18; \ldots$

(h) $3; 9; 15; 21; 27; \ldots$
(i) $10; 24; 44; 70; 102; \ldots$
(j) $1; 2{,}5; 5; 8{,}5; 13; \ldots$
(k) $2{,}5; 6; 10{,}5; 16; 22{,}5; \ldots$
(l) $0{,}5; 9; 20{,}5; 35; 52{,}5; \ldots$

3. Given $T_n = 2n^2$, find for which value of n, $T_n = 242$
4. Given $T_n = (n-4)^2$, find for which value of n, $T_n = 36$
5. Given $T_n = n^2 + 4$, find for which value of n, $T_n = 85$
6. Given $T_n = 3n^2$, find T_{11}
7. Given $T_n = 7n^2 + 4n$, find T_9
8. Given $T_n = 4n^2 + 3n - 1$, find T_5
9. Given $T_n = 1{,}5n^2$, find T_{10}
10. For each of the quadratic sequences, find the common second difference, the formula for the general term and then use the formula to find a_{100}.
    (a) $4; 7; 12; 19; 28; \ldots$
    (b) $2; 8; 18; 32; 50; \ldots$
    (c) $7; 13; 23; 37; 55; \ldots$
    (d) $5; 14; 29; 50; 77; \ldots$
    (e) $7; 22; 47; 82; 127; \ldots$
    (f) $3; 10; 21; 36; 55; \ldots$
    (g) $3; 7; 13; 21; 31; \ldots$
    (h) $3; 9; 17; 27; 39; \ldots$

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(1.) 0177 (2.) 0178 (3.) 0179 (4.) 017a (5.) 017b (6.) 017c
(7.) 017d (8.) 017e (9.) 017f (10.) 017g

Finance 6

6.1 Introduction

EMBR

In Grade 10, the concepts of simple and compound interest were introduced. Here we will extend those concepts, so it is a good idea to revise what you've learnt. After you have mastered the techniques in this chapter, you will understand depreciation and will learn how to determine which bank is offering the best interest rate.

See introductory video: VMemn at www.everythingmaths.co.za

6.2 Depreciation

EMBS

It is said that when you drive a new car out of the dealership, it loses 20% of its value, because it is now "second-hand". And from there on the value keeps falling, or *depreciating*. Second hand cars are cheaper than new cars, and the older the car, usually the cheaper it is. If you buy a second-hand (or should we say *pre-owned*!) car from a dealership, they will base the price on something called *book value*.

The book value of the car is the value of the car taking into account the loss in value due to wear, age and use. We call this loss in value *depreciation*, and in this section we will look at two ways of how this is calculated. Just like interest rates, the two methods of calculating depreciation are *simple* and *compound* methods.

The terminology used for simple depreciation is **straight-line depreciation** and for compound depreciation is **reducing-balance depreciation**. In the straight-line method the value of the asset is reduced by the same constant amount each year. In compound depreciation or reducing-balance the value of the asset is reduced by the same percentage each year. This means that the value of an asset does not decrease by a constant amount each year, but the decrease is most in the first year, then by a smaller amount in the second year and by an even smaller amount in the third year, and so on.

Extension:	*Depreciation*

You may be wondering why we need to calculate depreciation. Determining the value of assets (as in the example of the second hand cars) is one reason, but there is also a more financial reason for calculating depreciation — tax! Companies can take depreciation into account as an expense, and thereby reduce their taxable income. A lower taxable income means that the company will pay less income tax to the Revenue Service.

6.3 Simple Decay or Straight-line depreciation

EMBT

Let us return to the second-hand cars. One way of calculating a depreciation amount would be to assume that the car has a limited useful life. Simple depreciation assumes that the value of the car decreases by an equal amount each year. For example, let us say the limited useful life of a car is 5 years, and the cost of the car today is R60 000. What we are saying is that after 5 years you will have to buy a new car, which means that the old one will be valueless at that point in time. Therefore, the amount of depreciation is calculated:

$$\frac{\text{R60 000}}{\text{5 years}} = \text{R12 000} \quad \text{per year.}$$

The value of the car is then:

End of Year 1	R60 000 $-$ 1 $\times$ (R12 000)	= R48 000
End of Year 2	R60 000 $-$ 2 $\times$ (R12 000)	= R36 000
End of Year 3	R60 000 $-$ 3 $\times$ (R12 000)	= R24 000
End of Year 4	R60 000 $-$ 4 $\times$ (R12 000)	= R12 000
End of Year 5	R60 000 $-$ 5 $\times$ (R12 000)	= R0

This looks similar to the formula for simple interest:

$$\text{Total Interest after } n \text{ years} = n \times (P \times i)$$

where i is the annual percentage interest rate and P is the principal amount.

If we replace the word *interest* with the word *depreciation* and the word *principal* with the words *initial value* we can use the same formula:

$$\text{Total depreciation after } n \text{ years} = n \times (P \times i)$$

Then the book value of the asset after n years is:

$$\begin{aligned} \text{Initial Value – Total depreciation after } n \text{ years} &= P - n \times (P \times i) \\ A &= P(1 - n \times i) \end{aligned}$$

For example, the book value of the car after two years can be simply calculated as follows:

$$\begin{aligned} \text{Book Value after 2 years} &= P(1 - n \times i) \\ &= \text{R60 000}(1 - 2 \times 20\%) \\ &= \text{R60 000}(1 - 0{,}4) \\ &= \text{R60 000}(0{,}6) \\ &= \text{R36 000} \end{aligned}$$

as expected.

Note that the difference between the simple interest calculations and the simple decay calculations is that while the interest adds value to the principal amount, the depreciation amount reduces value!

Example 1: Simple Decay method

QUESTION

A car is worth R240 000 now. If it depreciates at a rate of 15% p.a. on a straight-line depreciation, what is it worth in 5 years' time?

SOLUTION

Step 1 : ***Determine what has been provided and what is required***

$$\begin{aligned} P &= R240\ 000 \\ i &= 0{,}15 \\ n &= 5 \\ A & \text{ is required} \end{aligned}$$

Step 2 : ***Determine how to approach the problem***

$$\begin{aligned} A &= P(1 - i \times n) \\ A &= 240\ 000(1 - (0{,}15 \times 5)) \end{aligned}$$

Step 3 : ***Solve the problem***

$$\begin{aligned} A &= 240\ 000(1 - 0{,}75) \\ &= 240\ 000 \times 0{,}25 \\ &= 60\ 000 \end{aligned}$$

Step 4 : ***Write the final answer***
In 5 years' time the car is worth R60 000

Example 2: Simple Decay

QUESTION

A small business buys a photocopier for R12 000. For the tax return the owner depreciates this asset over 3 years using a straight-line depreciation method. What amount will he fill in on *his tax form after 1 year, after 2 years and then after 3 years?*

SOLUTION

Step 1 : ***Understanding the question***
The owner of the business wants the photocopier to depreciate to R0 after 3 years. Thus, the value of the photocopier will go down by $12\ 000 \div 3 =$ R4 000 per year.

Step 2 : ***Value of the photocopier after 1 year***
$12\ 000 - 4\ 000 =$ R8 000

Step 3 : ***Value of the machine after 2 years***
$8\ 000 - 4\ 000 =$ R4 000

Step 4 : ***Write the final answer***
$4\ 000 - 4\ 000 = 0$
After 3 years the photocopier is worth nothing

Extension:	*Salvage Value*

Looking at the same example of our car with an initial value of R60 000, what if we suppose that we think we would be able to sell the car at the end of the 5 year period for R10 000? We call this amount the "Salvage Value".

We are still assuming simple depreciation over a useful life of 5 years, but now instead of depreciating the full value of the asset, we will take into account the salvage value, and will only apply the depreciation to the value of the asset that we expect not to recoup, i.e. R60 000−R10 000 =R50 000.

The annual depreciation amount is then calculated as (R60 000−R10 000)/5 =R10000

In general, the formula for simple (straight line) depreciation:

$$\text{Annual Depreciation} = \frac{\text{Initial Value} - \text{Salvage Value}}{\text{Useful Life}}$$

Exercise 6 - 1

1. A business buys a truck for R560 000. Over a period of 10 years the value of the truck depreciates to R0 (using the straight-line method). What is the value of the truck after 8 years?

2. Shrek wants to buy his grandpa's donkey for R800. His grandpa is quite pleased with the offer, seeing that it only depreciated at a rate of 3% per year using the straight-line method. Grandpa bought the donkey 5 years ago. What did grandpa pay for the donkey then?

3. Seven years ago, Rocco's drum kit cost him R12 500. It has now been valued at R2 300. What rate of simple depreciation does this represent?

4. Fiona buys a DStv satellite dish for R3 000. Due to weathering, its value depreciates simply at 15% per annum. After how long will the satellite dish be worth nothing?

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(1.) 017h (2.) 017i (3.) 017j (4.) 017k

6.4 Compound Decay or Reducing-balance depreciation

EMBU

The second method of calculating depreciation is to assume that the value of the asset decreases at a certain annual rate, but that the initial value of the asset this year, is the book value of the asset at the end of last year.

For example, if our second hand car has a limited useful life of 5 years and it has an initial value of R60 000, then the interest rate of depreciation is 20% (100%/5 years). After 1 year, the car is worth:

$$\begin{aligned}\text{Book Value after first year} &= P(1 - n \times i) \\ &= \text{R60 000}(1 - 1 \times 20\%) \\ &= \text{R60 000}(1 - 0{,}2) \\ &= \text{R60 000}(0{,}8) \\ &= \text{R48 000}\end{aligned}$$

At the beginning of the second year, the car is now worth R48 000, so after two years, the car is worth:

$$\begin{aligned}\text{Book Value after second year} &= P(1 - n \times i) \\ &= \text{R48 000}(1 - 1 \times 20\%) \\ &= \text{R48 000}(1 - 0{,}2) \\ &= \text{R48 000}(0{,}8) \\ &= \text{R38 400}\end{aligned}$$

We can tabulate these values.

End of first year	$\text{R60 000}(1 - 1 \times 20\%)$	$=\text{R60 000}(1 - 1 \times 20\%)^1$	= R48 000,00
End of second year	$\text{R48 000}(1 - 1 \times 20\%)$	$=\text{R60 000}(1 - 1 \times 20\%)^2$	= R38 400,00
End of third year	$\text{R38 400}(1 - 1 \times 20\%)$	$=\text{R60 000}(1 - 1 \times 20\%)^3$	= R30 720,00
End of fourth year	$\text{R30 720}(1 - 1 \times 20\%)$	$=\text{R60 000}(1 - 1 \times 20\%)^4$	= R24 576,00
End of fifth year	$\text{R24 576}(1 - 1 \times 20\%)$	$=\text{R60 000}(1 - 1 \times 20\%)^5$	= R19 608,80

We can now write a general formula for the book value of an asset if the depreciation is compounded.

$$\text{Initial Value – Total depreciation after } n \text{ years} = P(1 - i)^n \quad (6.1)$$

For example, the book value of the car after two years can be simply calculated as follows:

$$\begin{aligned}\text{Book Value after 2 years: } A &= P(1 - i)^n \\ &= \text{R60 000}(1 - 20\%)^2 \\ &= \text{R60 000}(1 - 0{,}2)^2 \\ &= \text{R60 000}(0{,}8)^2 \\ &= \text{R38 400}\end{aligned}$$

as expected.

Note that the difference between the compound interest calculations and the compound depreciation calculations is that while the interest adds value to the principal amount, the depreciation amount reduces value!

Example 3: Compound Depreciation

QUESTION

The flamingo population of the Berg river mouth is depreciating on a reducing balance at a rate of 12% p.a. If there are now 3 200 flamingos in the wetlands of the Berg river mouth, how many will there be in 5 years' time? Answer to three significant figures.

SOLUTION

Step 1 : ***Determine what has been provided and what is required***

$$\begin{aligned} P &= 3\ 200 \\ i &= 0{,}12 \\ n &= 5 \\ A &\ \text{is required} \end{aligned}$$

Step 2 : ***Determine how to approach the problem***

$$\begin{aligned} A &= P(1-i)^n \\ A &= 3\ 200(1-0{,}12)^5 \end{aligned}$$

Step 3 : ***Solve the problem***

$$\begin{aligned} A &= 3\ 200(0{,}88)^5 \\ &= 1688{,}742134 \end{aligned}$$

Step 4 : ***Write the final answer***

There would be approximately 1 690 flamingos in 5 years' time.

Example 4: Compound Depreciation

QUESTION

Farmer Brown buys a tractor for R250 000 which depreciates by 20% per year using the compound depreciation method. What is the depreciated value of the tractor after 5 years?

SOLUTION

Step 1 : ***Determine what has been provided and what is required***

$$\begin{aligned} P &= R250\ 000 \\ i &= 0{,}2 \\ n &= 5 \\ A &\ \text{is required} \end{aligned}$$

Step 2 : ***Determine how to approach the problem***

$$\begin{aligned} A &= P(1-i)^n \\ A &= 250\ 000(1-0{,}2)^5 \end{aligned}$$

Step 3 : ***Solve the problem***

$$\begin{aligned} A &= 250\ 000(0{,}8)^5 \\ &= 81\ 920 \end{aligned}$$

Step 4 : ***Write the final answer***
Depreciated value after 5 years is R81 920

Exercise 6 - 2

1. On January 1, 2008 the value of my Kia Sorento is R320 000. Each year after that, the cars value will decrease 20% of the previous years value. What is the value of the car on January 1, 2012?
2. The population of Bonduel decreases at a reducing-balance rate of $9{,}5\%$ per annum as people migrate to the cities. Calculate the decrease in population over a period of 5 years if the initial population was 2 178 000.
3. A $20\,\text{kg}$ watermelon consists of 98% water. If it is left outside in the sun it loses 3% of its water each day. How much does it weigh after a month of 31 days?
4. A computer depreciates at $x\%$ per annum using the reducing-balance method. Four years ago the value of the computer was R10 000 and is now worth R4 520. Calculate the value of x correct to two decimal places.

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(1.) 017m (2.) 017n (3.) 017p (4.) 017q

6.5 Present and Future Values of an Investment or Loan

EMBV

Now or Later

EMBW

When we studied simple and compound interest we looked at having a sum of money now, and calculating what it will be worth in the future. Whether the money was borrowed or invested, the calculations examined what the total money would be at some future date. We call these *future values*.

It is also possible, however, to look at a sum of money in the future, and work out what it is worth now. This is called a *present value*.

For example, if R1 000 is deposited into a bank account now, the future value is what that amount will accrue to by some specified future date. However, if R1 000 is needed at some future time, then the present value can be found by working backwards — in other words, how much must be invested to ensure the money grows to R1 000 at that future date?

The equation we have been using so far in compound interest, which relates the open balance (P), the closing balance (A), the interest rate (i as a rate per annum) and the term (n in years) is:

$$A = P\,.\,(1+i)^n \tag{6.2}$$

Using simple algebra, we can solve for P instead of A, and come up with:

$$P = A\,.\,(1+i)^{-n} \tag{6.3}$$

This can also be written as follows, but the first approach is usually preferred.

$$P = \frac{A}{(1+i)^n} \tag{6.4}$$

Now think about what is happening here. In Equation 6.2, we start off with a sum of money and we let it grow for n years. In Equation 6.3 we have a sum of money which we know in n years time, and we "unwind" the interest — in other words we take off interest for n years, until we see what it is worth right now.

We can test this as follows. If I have R1 000 now and I invest it at 10% for 5 years, I will have:

$$\begin{aligned} A &= P\,.\,(1+i)^n \\ &= \text{R1 000}(1+10\%)^5 \\ &= \text{R1 610,51} \end{aligned}$$

at the end. BUT, if I know I have to have R1610,51 in 5 years time, I need to invest:

$$\begin{aligned} P &= A\,.\,(1+i)^{-n} \\ &= \text{R1 610,51}(1+10\%)^{-5} \\ &= \text{R1 000} \end{aligned}$$

We end up with R1 000 which — if you think about it for a moment — is what we started off with. Do you see that?

Of course we could apply the same techniques to calculate a present value amount under simple interest rate assumptions — we just need to solve for the opening balance using the equations for simple interest.

$$A = P(1 + i \times n) \quad (6.5)$$

Solving for P gives:

$$P = \frac{A}{(1 + i \times n)} \quad (6.6)$$

Let us say you need to accumulate an amount of R1 210 in 3 years time, and a bank account pays *simple interest* of 7%. How much would you need to invest in this bank account today?

$$\begin{aligned} P &= \frac{A}{1 + n\,.\,i} \\ &= \frac{\text{R1 210}}{1 + 3 \times 7\%} \\ &= \text{R1 000} \end{aligned}$$

Does this look familiar? Look back to the simple interest worked example in Grade 10. There we started with an amount of R1 000 and looked at what it would grow to in 3 years' time using simple interest rates. Now we have worked backwards to see what amount we need as an opening balance in order to achieve the closing balance of R1 210.

In practise, however, present values are usually always calculated assuming compound interest. So unless you are explicitly asked to calculate a present value (or opening balance) using simple interest rates, make sure you use the compound interest rate formula!

Exercise 6 - 3

1. After a 20-year period Josh's lump sum investment matures to an amount of R313 550. How much did he invest if his money earned interest at a rate of $13{,}65\%$ p.a. compounded half yearly for the first 10 years, $8{,}4\%$ p.a. compounded quarterly for the next five years and $7{,}2\%$ p.a. compounded monthly for the remaining period?
2. A loan has to be returned in two equal semi-annual instalments. If the rate of interest is 16% per annum, compounded semi-annually and each instalment is R1 458, find the sum borrowed.

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(1.) 017r (2.) 017s

6.6 Finding i

EMBX

By this stage in your studies of the mathematics of finance, you have always known what interest rate to use in the calculations, and how long the investment or loan will last. You have then either taken a known starting point and calculated a future value, or taken a known future value and calculated a present value.

But here are other questions you might ask:

1. I want to borrow R2 500 from my neighbour, who said I could pay back R3 000 in 8 months time. What interest is she charging me?

2. I will need R450 for some university textbooks in 1,5 years time. I currently have R400. What interest rate do I need to earn to meet this goal?

Each time that you see something different from what you have seen before, start off with the basic equation that you should recognise very well:

$$A = P\,.\,(1+i)^n$$

If this were an algebra problem, and you were told to "solve for i", you should be able to show that:

$$\begin{aligned} \frac{A}{P} &= (1+i)^n \\ \sqrt[n]{\frac{A}{P}} &= 1+i \\ \sqrt[n]{\frac{A}{P}} - 1 &= i \\ \therefore i &= \sqrt[n]{\frac{A}{P}} - 1 \end{aligned}$$

You do not need to memorise this equation, it is easy to derive any time you need it!

So let us look at the two examples mentioned above.

1. Check that you agree that $P =$R2 500, $A =$R3 000, $n = \frac{8}{12} = \frac{2}{3}$. This means that:

$$\begin{aligned} i &= \sqrt[\frac{2}{3}]{\frac{3000}{2500}} - 1 \\ &= 0{,}314534... \\ &= 31{,}45\% \end{aligned}$$

   Ouch! That is not a very generous neighbour you have.

2. Check that $P =$R400, $A =$R450, $n = 1{,}5$

$$\begin{aligned} i &= \sqrt[1{,}5]{\frac{450}{400}} - 1 \\ &= 0{,}0816871... \\ &= 8{,}17\% \end{aligned}$$

   This means that as long as you can find a bank which pays more than $8{,}17\%$ interest, you should have the money you need!

Note that in both examples, we expressed n as a number of years ($\frac{8}{12}$ years, not 8 because that is the number of months) which means i is the annual interest rate. Always keep this in mind — keep years with years to avoid making silly mistakes.

Exercise 6 - 4

1. A machine costs R45 000 and has a scrap value of R9 000 after 10 years. Determine the annual rate of depreciation if it is calculated on the reducing balance method.
2. After 5 years an investment doubled in value. At what annual rate was interest compounded?

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(1.) 017t (2.) 017u

6.7 Finding n — Trial and Error

EMBY

By this stage you should be seeing a pattern. We have our standard formula, which has a number of variables:

$$A = P\,.\,(1+i)^n$$

We have solved for A (in Grade 10), P (in Section 6.5) and i (in Section 6.6). This time we are going to solve for n. In other words, if we know what the starting sum of money is and what it grows to, and if we know what interest rate applies — then we can work out how long the money needs to be invested for all those other numbers to tie up.

This section will calculate n by trial and error and by using a calculator. The proper algebraic solution will be learnt in Grade 12.

Solving for n, we can write:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{P} &= (1+i)^n \end{aligned}$$

Now we have to examine the numbers involved to try to determine what a possible value of n is. Refer to your Grade 10 notes for some ideas as to how to go about finding n.

Example 5: *Term of Investment — Trial and Error*

QUESTION

We invest R3 500 into a savings account which pays 7,5% compound interest for an unknown period of time, at the end of which our account is worth R4 044,69. How long did we invest the money?

SOLUTION

Step 1 : **Determine what is given and what is required**

- $P =$R3 500
- $i = 7{,}5\%$
- $A =$R4 044,69

We are required to find n.

Step 2 : **Determine how to approach the problem**

We know that:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{P} &= (1+i)^n \end{aligned}$$

Step 3 : ***Solve the problem***

$$\begin{aligned} \frac{\text{R4 044,69}}{\text{R3 500}} &= (1 + 7{,}5\%)^n \\ 1{,}156 &= (1{,}075)^n \end{aligned}$$

We now use our calculator and try a few values for n.

Possible n	$1{,}075^n$
1,0	1,075
1,5	1,115
2,0	1,156
2,5	1,198

We see that n is close to 2.

Step 4 : ***Write final answer***

The R3 500 was invested for about 2 years.

Exercise 6 - 5

1. A company buys two types of motor cars: The Acura costs R80 600 and the Brata R101 700, V.A.T. included. The Acura depreciates at a rate, compounded annually, of 15,3% per year and the Brata at 19,7%, also compounded annually, per year. After how many years will the book value of the two models be the same?
2. The fuel in the tank of a truck decreases every minute by 5,5% of the amount in the tank at that point in time. Calculate after how many minutes there will be less than 30 l in the tank if it originally held 200 l.

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(1.) 017v (2.) 017w

6.8 Nominal and Effective Interest Rates

EMBZ

So far we have discussed annual interest rates, where the interest is quoted as a per annum amount. Although it has not been explicitly stated, we have assumed that when the interest is quoted as a per annum amount it means that the interest is paid once a year.

Interest however, may be paid more than just once a year, for example we could receive interest on a monthly basis, i.e. 12 times per year. So how do we compare a monthly interest rate, say, to an annual interest rate? This brings us to the concept of the *effective annual interest rate*.

One way to compare different rates and methods of interest payments would be to compare the closing balances under the different options, for a given opening balance. Another, more widely used, way is to calculate and compare the *effective annual interest rate* on each option. This way, regardless of the differences in how frequently the interest is paid, we can compare apples-with-apples.

Tip

Remember, the trick to using the formulae is to define the time period, and use the interest rate relevant to the time period.

For example, a savings account with an opening balance of R1 000 offers a compound interest rate of 1% per month which is paid at the end of every month. We can calculate the accumulated balance at the end of the year using the formulae from the previous section. But be careful our interest rate has been given as a monthly rate, so we need to use the same units (months) for our time period of measurement.

So we can calculate the amount that would be accumulated by the end of 1-year as follows:

$$\begin{aligned}\text{Closing Balance after 12 months} &= P \times (1+i)^n \\ &= \text{R1 000} \times (1+1\%)^{12} \\ &= \text{R1 126,83}\end{aligned}$$

Note that because we are using a monthly time period, we have used $n = 12$ months to calculate the balance at the end of one year.

The effective annual interest rate is an annual interest rate which represents the equivalent per annum interest rate assuming compounding.

It is the annual interest rate in our Compound Interest equation that equates to the same accumulated balance after one year. So we need to solve for the effective annual interest rate so that the accumulated balance is equal to our calculated amount of R1 126,83.

We use i_{12} to denote the monthly interest rate. We have introduced this notation here to distinguish between the annual interest rate, i. Specifically, we need to solve for i in the following equation:

$$\begin{aligned}P \times (1+i)^1 &= P \times (1+i_{12})^{12} \\ (1+i) &= (1+i_{12})^{12} \quad \text{divide both sides by } P \\ i &= (1+i_{12})^{12} - 1 \quad \text{subtract 1 from both sides}\end{aligned}$$

For the example, this means that the effective annual rate for a monthly rate $i_{12} = 1\%$ is:

$$\begin{aligned}i &= (1+i_{12})^{12} - 1 \\ &= (1+1\%)^{12} - 1 \\ &= 0{,}12683 \\ &= 12{,}683\%\end{aligned}$$

If we recalculate the closing balance using this annual rate we get:

$$\begin{aligned}\text{Closing Balance after 1 year} &= P \times (1+i)^n \\ &= \text{R1 000} \times (1+12{,}683\%)^1 \\ &= \text{R1 126,83}\end{aligned}$$

which is the same as the answer obtained for 12 months.

Note that this is greater than simply multiplying the monthly rate by ($12 \times 1\% = 12\%$) due to the effects of compounding. The difference is due to interest on interest. We have seen this before, but it is an important point!

The General Formula

EMBAA

So we know how to convert a monthly interest rate into an effective annual interest. Similarly, we can convert a quarterly or semi-annual interest rate (or an interest rate of any frequency for that matter) into an effective annual interest rate.

For a quarterly interest rate of say 3% per quarter, the interest will be paid four times per year (every three months). We can calculate the effective annual interest rate by solving for i:

$$P(1+i) = P(1+i_4)^4$$

where i_4 is the quarterly interest rate.

So $(1+i) = (1{,}03)^4$, and so $i = 12{,}55\%$. This is the effective annual interest rate.

In general, for interest paid at a frequency of T times per annum, the follow equation holds:

$$P(1+i) = P(1+i_T)^T \tag{6.7}$$

where i_T is the interest rate paid T times per annum.

Decoding the Terminology

EMBAB

Market convention however, is not to state the interest rate as say 1% per month, but rather to express this amount as an annual amount which in this example would be paid monthly. This annual amount is called the nominal amount.

The market convention is to quote a nominal interest rate of "12% per annum paid monthly" instead of saying (an effective) 1% per month. We know from a previous example, that a nominal interest rate of 12% per annum paid monthly, equates to an effective annual interest rate of $12{,}68\%$, and the difference is due to the effects of interest-on-interest.

So if you are given an interest rate expressed as an annual rate but paid more frequently than annual, we first need to calculate the actual interest paid per period in order to calculate the effective annual interest rate.

$$\text{monthly interest rate} = \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}} \tag{6.8}$$

For example, the monthly interest rate on 12% interest per annum paid monthly, is:

$$\begin{aligned}\text{monthly interest rate} &= \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}} \\ &= \frac{12\%}{12 \text{ months}} \\ &= 1\% \text{ per month}\end{aligned}$$

The same principle applies to other frequencies of payment.

Example 6: *Nominal Interest Rate*

QUESTION

Consider a savings account which pays a nominal interest at 8% per annum, paid quarterly. Calculate (a) the interest amount that is paid each quarter, and (b) the effective annual interest rate.

SOLUTION

Step 1 : ***Determine what is given and what is required***

We are given that a savings account has a nominal interest rate of 8% paid quarterly. We are required to find:

- the quarterly interest rate, i_4
- the effective annual interest rate, i

Step 2 : ***Determine how to approach the problem***

We know that:

$$\text{quarterly interest rate} = \frac{\text{Nominal interest Rate per annum}}{\text{number of quarters per year}}$$

and

$$P(1+i) = P(1+i_T)^T$$

where T is 4 because there are 4 payments each year.

Step 3 : ***Calculate the monthly interest rate***

$$\begin{aligned}\text{quarterly interest rate} &= \frac{\text{Nominal interest rate per annum}}{\text{number of periods per year}} \\ &= \frac{8\%}{4 \text{ quarters}} \\ &= 2\% \text{ per quarter}\end{aligned}$$

Step 4 : ***Calculate the effective annual interest rate***

The effective annual interest rate (i) is calculated as:

$$\begin{aligned}(1+i) &= (1+i_4)^4 \\ (1+i) &= (1+2\%)^4 \\ i &= (1+2\%)^4 - 1 \\ &= 8{,}24\%\end{aligned}$$

Step 5 : ***Write the final answer***

The quarterly interest rate is 2% and the effective annual interest rate is 8,24%, for a nominal interest rate of 8% paid quarterly.

Example 7: *Nominal Interest Rate*

QUESTION

On their saving accounts, Echo Bank offers an interest rate of 18% nominal, paid monthly. If you save R100 in such an account now, how much would the amount have accumulated to in 3 years' time?

SOLUTION

Step 1 : ***Determine what is given and what is required***

Interest rate is 18% nominal paid monthly. There are 12 months in a year. We are working with a yearly time period, so $n = 3$. The amount we have saved is R100, so $P = 100$. We need the accumulated value, A.

Step 2 : ***Recall relevant formulae***

We know that

$$\text{monthly interest rate} = \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}}$$

for converting from nominal interest rate to effective interest rate, we have

$$1 + i = (1 + i_T)^T$$

and for calculating accumulated value, we have

$$A = P \times (1 + i)^n$$

Step 3 : ***Calculate the effective interest rate***

There are 12 month in a year, so

$$\begin{aligned} i_{12} &= \frac{\text{Nominal annual interest rate}}{12} \\ &= \frac{18\%}{12} \\ &= 1{,}5\% \text{ per month} \end{aligned}$$

and then, we have

$$\begin{aligned} 1 + i &= (1 + i_{12})^{12} \\ i &= (1 + i_{12})^{12} - 1 \\ &= (1 + 1{,}5\%)^{12} - 1 \\ &= (1{,}015)^{12} - 1 \\ &= 19{,}56\% \end{aligned}$$

Step 4 : ***Reach the final answer***

$$\begin{aligned} A &= P \times (1 + i)^n \\ &= 100 \times (1 + 19{,}56\%)^3 \\ &= 100 \times 1{,}7091 \\ &= 170{,}91 \end{aligned}$$

Step 5 : ***Write the final answer***

The accumulated value is R$170{,}91$. (Remember to round off to the the nearest cent.)

Exercise 6 - 6

1. Calculate the effective rate equivalent to a nominal interest rate of $8{,}75\%$ p.a. compounded monthly.

2. Cebela is quoted a nominal interest rate of $9{,}15\%$ per annum compounded every four months on her investment of R$85\ 000$. Calculate the effective rate per annum.

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(1.) 017x (2.) 017y

6.9 Formula Sheet

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As an easy reference, here are the key formulae that we derived and used during this chapter. While memorising them is nice (there are not many), it is the application that is useful. Financial experts are not paid a salary in order to recite formulae, they are paid a salary to use the right methods to solve financial problems.

Definitions

EMBAD

P	Principal (the amount of money at the starting point of the calculation)
i	interest rate, normally the effective rate per annum
n	period for which the investment is made
i_T	the interest rate paid T times per annum, i.e. $i_T = \frac{\text{Nominal Interest Rate}}{T}$

Equations

EMBAE

$$\begin{aligned} Simple\ Increase : A &= P(1 + i \times n) \\ Compound\ Increase : A &= P(1 + i)^n \\ Simple\ Decay : A &= P(1 - i \times n) \\ Compound\ Decay : A &= P(1 - i)^n \\ Effective\ Annual\ Interest\ Rate(i) : (1 + i) &= (1 + i_T)^T \end{aligned}$$

Chapter 6 End of Chapter Exercises

1. Shrek buys a Mercedes worth R385 000 in 2007. What will the value of the Mercedes be at the end of 2013 if:
   (a) the car depreciates at 6% p.a. straight-line depreciation
   (b) the car depreciates at 12% p.a. reducing-balance depreciation.
2. Greg enters into a 5-year hire-purchase agreement to buy a computer for R8 900. The interest rate is quoted as 11% per annum based on simple interest. Calculate the required monthly payment for this contract.
3. A computer is purchased for R16000. It depreciates at 15% per annum.
   (a) Determine the book value of the computer after 3 years if depreciation is calculated according to the straight-line method.
   (b) Find the rate, according to the reducing-balance method, that would yield the same book value as in 3a) after 3 years.
4. Maggie invests R12 500,00 for 5 years at 12% per annum compounded monthly for the first 2 years and 14% per annum compounded semi-annually for the next 3 years. How much will Maggie receive in total after 5 years?
5. Tintin invests R120000. He is quoted a nominal interest rate of 7,2% per annum compounded monthly.
   (a) Calculate the effective rate per annum correct to *three* decimal places.
   (b) Use the effective rate to calculate the value of Tintin's investment if he invested the money for 3 years.
   (c) Suppose Tintin invests his money for a total period of 4 years, but after 18 months makes a withdrawal of R20 000, how much will he receive at the end of the 4 years?
6. Paris opens accounts at a number of clothing stores and spends freely. She gets herself into terrible debt and she cannot pay off her accounts. She owes Hilton Fashion world R5 000 and the shop agrees to let Paris pay the bill at a nominal interest rate of 24% compounded monthly.
   (a) How much money will she owe Hilton Fashion World after two years?
   (b) What is the effective rate of interest that Hilton Fashion World is charging her?

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(1.) 017z (2.) 0180 (3.) 0181 (4.) 0182 (5.) 0183 (6.) 0184

Solving Quadratic Equations 7

7.1 Introduction

EMBAF

In Grade 10, the basics of solving linear equations, quadratic equations, exponential equations and linear inequalities were studied. This chapter extends that work by looking at different methods for solving quadratic equations.

See introductory video: VMemp at www.everythingmaths.co.za

7.2 Solution by Factorisation

EMBAG

How to solve quadratic equations by factorisation was discussed in Grade 10. Here is an example to remind you of what is involved.

Example 1: *Solution of Quadratic Equations*

QUESTION

Solve the equation $2x^2 - 5x - 12 = 0$.

SOLUTION

Step 1 : ***Determine whether the equation has common factors***
This equation has no common factors.

Step 2 : ***Determine if the equation is in the form*** $ax^2 + bx + c$ ***with*** $a > 0$
The equation is in the required form, with $a = 2$, $b = -5$ and $c = -12$.

Step 3 : ***Factorise the quadratic***
$2x^2 - 5x - 12$ has factors of the form:

$$(2x + s)(x + v)$$

with s and v constants to be determined. This multiplies out to

$$2x^2 + (s + 2v)x + sv$$

We see that $sv = -12$ and $s + 2v = -5$. This is a set of simultaneous equations in s and v, but it is easy to solve numerically. All the options for s and v are considered below.

s	v	$s + 2v$
2	-6	-10
-2	6	10
3	-4	-5
-3	4	5
4	-3	-2
-4	3	2
6	-2	2
-6	2	-2

We see that the combination $s = 3$ and $v = -4$ gives $s + 2v = -5$.

Step 4 : ***Write the equation with factors***

$$(2x + 3)(x - 4) = 0$$

Step 5 : ***Solve the equation***

If two brackets are multiplied together and give 0, then one of the brackets must be 0, therefore

$$2x + 3 = 0$$

or

$$x - 4 = 0$$

Therefore, $x = -\frac{3}{2}$ or $x = 4$

Step 6 : ***Write the final answer***

The solutions to $2x^2 - 5x - 12 = 0$ are $x = -\frac{3}{2}$ or $x = 4$.

It is important to remember that a quadratic equation has to be in the form $ax^2 + bx + c = 0$ before one can solve it using the factorisation method.

Example 2: *Solving quadratic equation by factorisation*

QUESTION

Solve for a: $a(a - 3) = 10$

SOLUTION

Step 1 : ***Rewrite the equation in the form*** $ax^2 + bx + c = 0$

Remove the brackets and move all terms to one side.

$$a^2 - 3a - 10 = 0$$

Step 2 : ***Factorise the trinomial***

$$(a+2)(a-5) = 0$$

Step 3 : ***Solve the equation***

$$a + 2 = 0$$

or

$$a - 5 = 0$$

Solve the two linear equations and check the solutions in the original equation.

Step 4 : ***Write the final answer***

Therefore, $a = -2$ or $a = 5$

Example 3: *Solving fractions that lead to a quadratic equation*

QUESTION

Solve for b: $\frac{3b}{b+2} + 1 = \frac{4}{b+1}$

SOLUTION

Step 1 : ***Multiply both sides over the lowest common denominator***

$$\frac{3b(b+1) + (b+2)(b+1)}{(b+2)(b+1)} = \frac{4(b+2)}{(b+2)(b+1)}$$

Step 2 : ***Determine the restrictions***

The restrictions are the values for b that would result in the denominator being 0. Since a denominator of 0 would make the fraction undefined, b cannot be these values. Therefore, $b \neq -2$ and $b \neq -1$

Step 3 : ***Simplify equation to the standard form***

The denominators on both sides of the equation are equal. This means we can drop them (by multiplying both sides of the equation by $(b+2)(b+1)$) and just work with the numerators.

$$\begin{aligned} 3b^2 + 3b + b^2 + 3b + 2 &= 4b + 8 \\ 4b^2 + 2b - 6 &= 0 \\ 2b^2 + b - 3 &= 0 \end{aligned}$$

Step 4 : ***Factorise the trinomial and solve the equation***

$$\begin{aligned} (2b+3)(b-1) &= 0 \\ 2b+3=0 \quad &or \quad b-1=0 \\ b=\frac{-3}{2} \quad &or \quad b=1 \end{aligned}$$

Step 5 : ***Check solutions in original equation***
Both solutions are valid
Therefore, $b=\frac{-3}{2}$ or $b=1$

Exercise 7 - 1

Solve the following quadratic equations by factorisation. Some answers may be left in surd form.

1. $2y^2-61=101$
2. $2y^2-10=0$
3. $y^2-4=10$
4. $2y^2-8=28$
5. $7y^2=28$
6. $y^2+28=100$
7. $7y^2+14y=0$
8. $12y^2+24y+12=0$
9. $16y^2-400=0$
10. $y^2-5y+6=0$
11. $y^2+5y-36=0$
12. $y^2+2y=8$
13. $-y^2-11y-24=0$
14. $13y-42=y^2$
15. $y^2+9y+14=0$
16. $y^2-5ky+4k^2=0$
17. $y(2y+1)=15$
18. $\frac{5y}{y-2}+\frac{3}{y}+2=\frac{-6}{y^2-2y}$
19. $\frac{y-2}{y+1}=\frac{2y+1}{y-7}$

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(1.) 0185 (2.) 0186 (3.) 0187 (4.) 0188 (5.) 0189 (6.) 018a
(7.) 018b (8.) 018c (9.) 018d (10.) 018e (11.) 018f (12.) 018g
(13.) 018h (14.) 018i (15.) 018j (16.) 018k (17.) 018m (18.) 018n
(19.) 018p

7.3 Solution by Completing the Square

EMBAH

We have seen that expressions of the form:

$$a^2x^2 - b^2$$

are known as differences of squares and can be factorised as follows:

$$(ax - b)(ax + b).$$

This simple factorisation leads to another technique to solve quadratic equations known as *completing the square*.

We demonstrate with a simple example, by trying to solve for x in:

$$x^2 - 2x - 1 = 0. \tag{7.1}$$

We cannot easily find factors of this term, but the first two terms look similar to the first two terms of the perfect square:

$$(x - 1)^2 = x^2 - 2x + 1.$$

However, we can cheat and create a perfect square by adding 2 to both sides of the equation in (7.1) as:

$$\begin{aligned} x^2 - 2x - 1 &= 0 \\ x^2 - 2x - 1 + 2 &= 0 + 2 \\ x^2 - 2x + 1 &= 2 \\ (x - 1)^2 &= 2 \\ (x - 1)^2 - 2 &= 0 \end{aligned}$$

Now we know that:

$$2 = (\sqrt{2})^2$$

which means that:

$$(x - 1)^2 - 2$$

is a difference of squares. Therefore we can write:

$$(x - 1)^2 - 2 = [(x - 1) - \sqrt{2}][(x - 1) + \sqrt{2}] = 0.$$

The solution to $x^2 - 2x - 1 = 0$ is then:

$$(x - 1) - \sqrt{2} = 0$$

or

$$(x - 1) + \sqrt{2} = 0.$$

This means $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$. This example demonstrates the use of *completing the square* to solve a quadratic equation.

Method: Solving Quadratic Equations by Completing the Square

1. Write the equation in the form $ax^2 + bx + c = 0$. e.g. $x^2 + 2x - 3 = 0$
2. Take the constant over to the right hand side of the equation, e.g. $x^2 + 2x = 3$
3. Make the coefficient of the x^2 term $= 1$, by dividing through by the existing coefficient.
4. Take half the coefficient of the x term, square it and add it to both sides of the equation, e.g. in $x^2 + 2x = 3$, half of the coefficient of the x term is 1 and $1^2 = 1$. Therefore we add 1 to both sides to get: $x^2 + 2x + 1 = 3 + 1$.

5. Write the left hand side as a perfect square: $(x+1)^2 - 4 = 0$
6. You should then be able to factorise the equation in terms of difference of squares and then solve for x:

$$\begin{aligned} [(x+1)-2][(x+1)+2)] &= 0 \\ (x-1)(x+3) &= 0 \\ \therefore x = 1 \quad \text{or} \quad x &= -3 \end{aligned}$$

Example 4: *Solving Quadratic Equations by Completing the Square*

QUESTION

Solve by completing the square:

$$x^2 - 10x - 11 = 0$$

SOLUTION

Step 1 : ***Write the equation in the form*** $ax^2 + bx + c = 0$

$$x^2 - 10x - 11 = 0$$

Step 2 : ***Take the constant over to the right hand side of the equation***

$$x^2 - 10x = 11$$

Step 3 : ***Check that the coefficient of the*** x^2 ***term is*** 1**.**
The coefficient of the x^2 term is 1.

Step 4 : ***Take half the coefficient of the*** x ***term, square it and add it to both sides***
The coefficient of the x term is -10. Therefore, half of the coefficient of the x term will be $\frac{(-10)}{2} = -5$ and the square of it will be $(-5)^2 = 25$. Therefore:

$$x^2 - 10x + 25 = 11 + 25$$

Step 5 : ***Write the left hand side as a perfect square***

$$(x-5)^2 - 36 = 0$$

Step 6 : ***Factorise equation as difference of squares***

$$(x-5)^2 - 36 = 0$$
$$[(x-5)+6][(x-5)-6] = 0$$

Step 7 : ***Solve for the unknown value***

$$(x+1)(x-11) = 0$$
$$\therefore x = -1 \quad \text{or} \quad x = 11$$

Example 5: *Solving Quadratic Equations by Completing the Square*

QUESTION

Solve by completing the square:

$$2x^2 - 8x - 16 = 0$$

SOLUTION

Step 1 : ***Write the equation in the form*** $ax^2 + bx + c = 0$

$$2x^2 - 8x - 16 = 0$$

Step 2 : ***Take the constant over to the right hand side of the equation***

$$2x^2 - 8x = 16$$

Step 3 : ***Check that the coefficient of the*** x^2 ***term is*** 1.

The coefficient of the x^2 term is 2. Therefore, divide both sides by 2:

$$x^2 - 4x = 8$$

Step 4 : ***Take half the coefficient of the*** x ***term, square it and add it to both sides***

The coefficient of the x term is -4; $\frac{(-4)}{2} = -2$ and $(-2)^2 = 4$. Therefore:

$$x^2 - 4x + 4 = 8 + 4$$

Step 5 : ***Write the left hand side as a perfect square***

$$(x-2)^2 - 12 = 0$$

Step 6 : ***Factorise equation as difference of squares***

$$[(x-2) + \sqrt{12}][(x-2) - \sqrt{12}] = 0$$

Step 7 : ***Solve for the unknown value***

$$[x - 2 + \sqrt{12}][x - 2 - \sqrt{12}] = 0$$
$$\therefore x = 2 - \sqrt{12} \quad \text{or} \quad x = 2 + \sqrt{12}$$

Step 8 : ***The last three steps can also be done in a different way***

Leave the left hand side written as a perfect square

$$(x-2)^2 = 12$$

Step 9 : ***Take the square root on both sides of the equation***

$$x-2 = \pm\sqrt{12}$$

Step 10 : ***Solve for*** x

Therefore $x = 2-\sqrt{12}$ or $x = 2+\sqrt{12}$

Compare to answer in step 7.

See video: VMeyf at www.everythingmaths.co.za

Exercise 7 - 2

Solve the following equations by completing the square:

1. $x^2 + 10x - 2 = 0$
2. $x^2 + 4x + 3 = 0$
3. $x^2 + 8x - 5 = 0$
4. $2x^2 + 12x + 4 = 0$
5. $x^2 + 5x + 9 = 0$
6. $x^2 + 16x + 10 = 0$
7. $3x^2 + 6x - 2 = 0$
8. $z^2 + 8z - 6 = 0$
9. $2z^2 - 11z = 0$
10. $5 + 4z - z^2 = 0$

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(1.) 018q (2.) 018r (3.) 018s (4.) 018t (5.) 018u (6.) 018v
(7.) 018w (8.) 018x (9.) 018y (10.) 018z

7.4 Solution by the Quadratic Formula

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It is not always possible to solve a quadratic equation by factorising and sometimes it is lengthy and tedious to solve a quadratic equation by completing the square. In these situations, you can use the *quadratic formula* that gives the solutions to any quadratic equation.

Consider the general form of the quadratic function:

$$f(x) = ax^2 + bx + c.$$

Factor out the a to get:

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right). \tag{7.2}$$

Now we need to do some detective work to figure out how to turn (7.2) into a perfect square plus some extra terms. We know that for a perfect square:

$$(m+n)^2 = m^2 + 2mn + n^2$$

and

$$(m-n)^2 = m^2 - 2mn + n^2$$

The key is the middle term on the right hand side, which is $2\times$ the first term $\times$ the second term of the left hand side. In (7.2), we know that the first term is x so $2\times$ the second term is $\frac{b}{a}$. This means that the second term is $\frac{b}{2a}$. So,

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2.$$

In general if you add a quantity and subtract the same quantity, nothing has changed. This means if we add and subtract $\left(\frac{b}{2a}\right)^2$ from the right hand side of Equation (7.2) we will get:

$$\begin{aligned}
f(x) &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &(7.3)\\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) &(7.4)\\
&= a\left(\left[x + \left(\frac{b}{2a}\right)\right]^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) &(7.5)\\
&= a\left[x + \left(\frac{b}{2a}\right)\right]^2 - \frac{b^2}{4a} + c &(7.6)
\end{aligned}$$

We set $f(x) = 0$ to find its roots, which yields:

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c \tag{7.7}$$

Now dividing by a and taking the square root of both sides gives the expression

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \tag{7.8}$$

Finally, solving for x implies that

$$\begin{aligned}
x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\\
&= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}
\end{aligned}$$

which can be further simplified to:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7.9}$$

These are the solutions to the quadratic equation. Notice that there are two solutions in general, but these may not always exists (depending on the sign of the expression $b^2 - 4ac$ under the square root). These solutions are also called the *roots* of the quadratic equation.

Example 6: *Using the quadratic formula*

QUESTION

Find the roots of the function $f(x) = 2x^2 + 3x - 7$.

SOLUTION

Step 1 : ***Determine whether the equation can be factorised***
The expression cannot be factorised. Therefore, the general quadratic formula must be used.

Step 2 : ***Identify the coefficients in the equation for use in the formula***
From the equation:

$$a = 2$$
$$b = 3$$
$$c = -7$$

Step 3 : ***Apply the quadratic formula***
Always write down the formula first and then substitute the values of a, b and c.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (7.10)$$
$$= \frac{-(3) \pm \sqrt{(3)^2 - 4(2)(-7)}}{2(2)} \quad (7.11)$$
$$= \frac{-3 \pm \sqrt{65}}{4} \quad (7.12)$$
$$= \frac{-3 \pm \sqrt{65}}{4} \quad (7.13)$$

Step 4 : ***Write the final answer***
The two roots of $f(x) = 2x^2 + 3x - 7$ are $x = \frac{-3+\sqrt{65}}{4}$ and $\frac{-3-\sqrt{65}}{4}$.

Example 7: *Using the quadratic formula but no solution*

QUESTION

Find the solutions to the quadratic equation $x^2 - 5x + 8 = 0$.

SOLUTION

Step 1 : ***Determine whether the equation can be factorised***
The expression cannot be factorised. Therefore, the general quadratic formula must be used.

Step 2 : ***Identify the coefficients in the equation for use in the formula***
From the equation:

$$a = 1$$

$$b = -5$$

$$c = 8$$

Step 3 : ***Apply the quadratic formula***

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (7.14)$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(8)}}{2(1)} \quad (7.15)$$

$$= \frac{5 \pm \sqrt{-7}}{2} \quad (7.16)$$

(7.17)

Step 4 : ***Write the final answer***
Since the expression under the square root is negative these are not real solutions ($\sqrt{-7}$ is not a real number). Therefore there are no real solutions to the quadratic equation $x^2 - 5x + 8 = 0$. This means that the graph of the quadratic function $f(x) = x^2 - 5x + 8$ has no x-intercepts, but that the entire graph lies above the x-axis.

See video: VMezc at www.everythingmaths.co.za

Exercise 7 - 3

Solve for t using the quadratic formula.

1. $3t^2 + t - 4 = 0$
2. $t^2 - 5t + 9 = 0$
3. $2t^2 + 6t + 5 = 0$
4. $4t^2 + 2t + 2 = 0$
5. $-3t^2 + 5t - 8 = 0$
6. $-5t^2 + 3t - 3 = 0$
7. $t^2 - 4t + 2 = 0$

8. $9t^2 - 7t - 9 = 0$
9. $2t^2 + 3t + 2 = 0$
10. $t^2 + t + 1 = 0$

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(1.) 0190 (2.) 0191 (3.) 0192 (4.) 0193 (5.) 0194 (6.) 0195
(7.) 0196 (8.) 0197 (9.) 0198 (10.) 0199

Tip

- In all the examples done so far, the solutions were left in surd form. Answers can also be given in decimal form, using the calculator. Read the instructions when answering questions in a test or exam whether to leave answers in surd form, or in decimal form to an appropriate number of decimal places.
- Completing the square as a method to solve a quadratic equation is only done when specifically asked.

Exercise 7 - 4

Solve the quadratic equations by either factorisation, completing the square or by using the quadratic formula:

- Always try to factorise first, then use the formula if the trinomial cannot be factorised.
- Do some of them by completing the square and then compare answers to those done using the other methods.

1. $24y^2 + 61y - 8 = 0$
2. $-8y^2 - 16y + 42 = 0$
3. $-9y^2 + 24y - 12 = 0$
4. $-5y^2 + 0y + 5 = 0$
5. $-3y^2 + 15y - 12 = 0$
6. $49y^2 + 0y - 25 = 0$
7. $-12y^2 + 66y - 72 = 0$
8. $-40y^2 + 58y - 12 = 0$
9. $-24y^2 + 37y + 72 = 0$
10. $6y^2 + 7y - 24 = 0$
11. $2y^2 - 5y - 3 = 0$
12. $-18y^2 - 55y - 25 = 0$
13. $-25y^2 + 25y - 4 = 0$
14. $-32y^2 + 24y + 8 = 0$
15. $9y^2 - 13y - 10 = 0$
16. $35y^2 - 8y - 3 = 0$
17. $-81y^2 - 99y - 18 = 0$
18. $14y^2 - 81y + 81 = 0$
19. $-4y^2 - 41y - 45 = 0$
20. $16y^2 + 20y - 36 = 0$
21. $42y^2 + 104y + 64 = 0$
22. $9y^2 - 76y + 32 = 0$
23. $-54y^2 + 21y + 3 = 0$
24. $36y^2 + 44y + 8 = 0$
25. $64y^2 + 96y + 36 = 0$
26. $12y^2 - 22y - 14 = 0$
27. $16y^2 + 0y - 81 = 0$
28. $3y^2 + 10y - 48 = 0$
29. $-4y^2 + 8y - 3 = 0$
30. $-5y^2 - 26y + 63 = 0$
31. $x^2 - 70 = 11$
32. $2x^2 - 30 = 2$
33. $x^2 - 16 = 2 - x^2$
34. $2y^2 - 98 = 0$
35. $5y^2 - 10 = 115$
36. $5y^2 - 5 = 19 - y^2$

More practice video solutions or help at www.everythingmaths.co.za

(1.) 01zs (2.) 01zt (3.) 01zu (4.) 01zv (5.) 01zw (6.) 01zx
(7.) 01zy (8.) 01zz (9.) 0200 (10.) 0201 (11.) 0202 (12.) 0203
(13.) 0204 (14.) 0205 (15.) 0206 (16.) 0207 (17.) 0208 (18.) 0209
(19.) 020a (20.) 020b (21.) 020c (22.) 020d (23.) 020e (24.) 020f
(25.) 020g (26.) 020h (27.) 020i (28.) 020j (29.) 020k (30.) 020m
(31.) 020n (32.) 020p (33.) 020q (34.) 020r (35.) 020s (36.) 020t

7.5 Finding an Equation When You Know its Roots

EMBAJ

We have mentioned before that the *roots* of a quadratic equation are the solutions or answers you get from solving the quadratic equation. Working back from the answers, will take you to an equation.

Example 8: *Find an equation when roots are given*

QUESTION

Find an equation with roots 13 *and* -5

SOLUTION

Step 1 : ***Write down as the product of two brackets***

The step before giving the solutions would be:

$$(x-13)(x+5)=0$$

Notice that the signs in the brackets are opposite of the given roots.

Step 2 : ***Remove brackets***

$$x^2-8x-65=0$$

Of course, there would be other possibilities as well when each term on each side of the *equals sign* is multiplied by a constant.

Example 9: *Fraction roots*

QUESTION

Find an equation with roots $-\frac{3}{2}$ *and* 4

SOLUTION

Step 1 : ***Product of two brackets***
Notice that if $x = -\frac{3}{2}$ then $2x + 3 = 0$
Therefore the two brackets will be:

$$(2x + 3)(x - 4) = 0$$

Step 2 : ***Remove brackets***
The equation is:

$$2x^2 - 5x - 12 = 0$$

Extension:	*Theory of Quadratic Equations - Advanced*

This section is not in the syllabus, but it gives one a good understanding about some of the solutions of the quadratic equations.

What is the Discriminant of a Quadratic Equation?

EMBAK

Consider a general quadratic function of the form $f(x) = ax^2 + bx + c$. The *discriminant* is defined as:

$$\Delta = b^2 - 4ac. \tag{7.18}$$

This is the expression under the square root in the formula for the roots of this function. We have already seen that whether the roots exist or not depends on whether this factor Δ is negative or positive.

The Nature of the Roots

EMBAL

Real Roots ($\Delta \geq 0$)

Consider $\Delta \geq 0$ for some quadratic function $f(x) = ax^2 + bx + c$. In this case there are solutions to the equation $f(x) = 0$ given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a} \tag{7.19}$$

If the expression under the square root is non-negative then the square root exists. These are the roots of the function $f(x)$.

There various possibilities are summarised in the figure below.

- Δ
  - $\Delta < 0$ **: imaginary roots**
  - $\Delta \geq 0$ **: real roots**
    - $\Delta > 0$ unequal roots
      - Δ a perfect square : rational roots
      - Δ not a perfect square : irrational roots
    - $\Delta = 0$ equal roots

Equal Roots ($\Delta = 0$)

If $\Delta = 0$, then the roots are equal and, from the formula, these are given by

$$x = -\frac{b}{2a} \tag{7.20}$$

Unequal Roots ($\Delta > 0$)

There will be two unequal roots if $\Delta > 0$. The roots of $f(x)$ are **rational** if Δ is a perfect square (a number which is the square of a rational number), since, in this case, $\sqrt{\Delta}$ is rational. Otherwise, if Δ is not a perfect square, then the roots are **irrational**.

Imaginary Roots ($\Delta < 0$)

If $\Delta < 0$, then the solution to $f(x) = ax^2 + bx + c = 0$ contains the square root of a negative number and therefore there are no real solutions. We therefore say that the roots of $f(x)$ are *imaginary* (the graph of the function $f(x)$ does not intersect the x-axis).

See video: VMfba at www.everythingmaths.co.za

Extension: *Theory of Quadratics - advanced exercises*

Exercise 7 - 5

1. [IEB, Nov. 2001, HG] Given: $x^2 + bx - 2 + k(x^2 + 3x + 2) = 0$, $(k \neq -1)$
   (a) Show that the discriminant is given by:
   $$\Delta = k^2 + 6bk + b^2 + 8$$
   (b) If $b = 0$, discuss the nature of the roots of the equation.
   (c) If $b = 2$, find the value(s) of k for which the roots are equal.
2. [IEB, Nov. 2002, HG] Show that $k^2x^2 + 2 = kx - x^2$ has non-real roots for all real values for k.
3. [IEB, Nov. 2003, HG] The equation $x^2 + 12x = 3kx^2 + 2$ has real roots.
   (a) Find the largest integral value of k.
   (b) Find one rational value of k, for which the above equation has rational roots.
4. [IEB, Nov. 2003, HG] In the quadratic equation $px^2 + qx + r = 0$, p, q and r are positive real numbers and form a geometric sequence. Discuss the nature of the roots.

5. [IEB, Nov. 2004, HG] Consider the equation:

$$k = \frac{x^2 - 4}{2x - 5} \qquad \text{where } x \neq \tfrac{5}{2}$$

   (a) Find a value of k for which the roots are equal.
   (b) Find an integer k for which the roots of the equation will be rational and unequal.

6. [IEB, Nov. 2005, HG]
   (a) Prove that the roots of the equation $x^2 - (a + b)x + ab - p^2 = 0$ are real for all real values of a, b and p.
   (b) When will the roots of the equation be equal?

7. [IEB, Nov. 2005, HG] If b and c can take on only the values 1; 2 or 3, determine all pairs (b; c) such that $x^2 + bx + c = 0$ has real roots.

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(1.) 019a (2.) 019b (3.) 019c (4.) 019d (5.) 019e (6.) 019f (7.) 019g

Chapter 7 End of Chapter Exercises

1. Solve: $x^2 - x - 1 = 0$ (Give your answer correct to two decimal places.)
2. Solve: $16(x + 1) = x^2(x + 1)$
3. Solve: $y^2 + 3 + \frac{12}{y^2 + 3} = 7$ (Hint: Let $y^2 + 3 = k$ and solve for k first and use the answer to solve y.)
4. Solve for x: $2x^4 - 5x^2 - 12 = 0$
5. Solve for x:
   (a) $x(x - 9) + 14 = 0$
   (b) $x^2 - x = 3$ (Show your answer correct to *one* decimal place.)
   (c) $x + 2 = \frac{6}{x}$ (correct to two decimal places)
   (d) $\frac{1}{x + 1} + \frac{2x}{x - 1} = 1$
6. Solve for x in terms of p by completing the square: $x^2 - px - 4 = 0$
7. The equation $ax^2 + bx + c = 0$ has roots $x = \frac{2}{3}$ and $x = -4$. Find one set of possible values for a, b and c.
8. The two roots of the equation $4x^2 + px - 9 = 0$ differ by 5. Calculate the value of p.
9. An equation of the form $x^2 + bx + c = 0$ is written on the board. Saskia and Sven copy it down incorrectly. Saskia has a mistake in the constant term and obtains the solutions -4 and 2. Sven has a mistake in the coefficient of x and obtains the solutions 1 and -15. Determine the correct equation that was on the board.
10. Bjorn stumbled across the following formula to solve the quadratic equation $ax^2 + bx + c = 0$ in a foreign textbook.

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

(a) Use this formula to solve the equation:

$$2x^2 + x - 3 = 0$$

(b) Solve the equation again, using factorisation, to see if the formula works for this equation.

(c) Trying to derive this formula to prove that it always works, Bjorn got stuck along the way. His attempt his shown below:

$$\begin{aligned}
ax^2 + bx + c &= 0 && \\
a + \frac{b}{x} + \frac{c}{x^2} &= 0 && \text{Divided by } x^2 \text{ where } x \neq 0 \\
\frac{c}{x^2} + \frac{b}{x} + a &= 0 && \text{Rearranged} \\
\frac{1}{x^2} + \frac{b}{cx} + \frac{a}{c} &= 0 && \text{Divided by } c \text{ where } c \neq 0 \\
\frac{1}{x^2} + \frac{b}{cx} &= -\frac{a}{c} && \text{Subtracted } \frac{a}{c} \text{ from both sides} \\
\therefore \frac{1}{x^2} + \frac{b}{cx} &+ \ldots && \text{Got stuck}
\end{aligned}$$

Complete his derivation.

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(1.) 019h (2.) 019i (3.) 019j (4.) 019k (5.) 019m (6.) 019n
(7.) 019p (8.) 019q (9.) 019r (10.) 019s

Solving Quadratic Inequalities 8

8.1 Introduction

EMBAM

Now that you know how to solve quadratic equations, you are ready to move on to solving quadratic inequalities. As with linear inequalities (which were covered in Grade 10) your solutions will be intervals on the number line, rather than single numbers.

See introductory video: VMfdy at www.everythingmaths.co.za

8.2 Quadratic Inequalities

EMBAN

A *quadratic inequality* is an inequality in one of the following forms:

$$ax^2 + bx + c > 0$$
$$ax^2 + bx + c \geq 0$$
$$ax^2 + bx + c < 0$$
$$ax^2 + bx + c \leq 0$$

Solving a quadratic inequality corresponds to working out in what region the graph of a quadratic function lies above or below the x-axis.

Example 1: *Quadratic Inequality*

QUESTION

Solve the inequality $4x^2 - 4x + 1 \leq 0$ and interpret the solution graphically.

SOLUTION

Step 1 : ***Factorise the quadratic***

Let $f(x) = 4x^2 - 4x + 1$. Factorising this quadratic function gives $f(x) = (2x - 1)^2$.

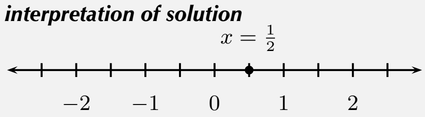
Step 2 : ***Re-write the original equation with factors***

$$(2x-1)^2 \leq 0$$

Step 3 : ***Solve the equation***
$f(x) = 0$ only when $x = \frac{1}{2}$.

Step 4 : ***Write the final answer***
This means that the graph of $f(x) = 4x^2 - 4x + 1$ touches the x-axis at $x = \frac{1}{2}$, but there are no regions where the graph is below the x-axis.

Step 5 : ***Graphical interpretation of solution***



Example 2: *Solving Quadratic Inequalities*

QUESTION

Find all the solutions to the inequality $x^2 - 5x + 6 \geq 0$.

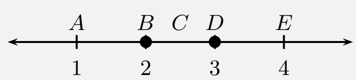
SOLUTION

Step 1 : ***Factorise the quadratic***
The factors of $x^2 - 5x + 6$ are $(x-3)(x-2)$.

Step 2 : ***Write the inequality with the factors***

$$\begin{aligned} x^2 - 5x + 6 &\geq 0 \\ (x-3)(x-2) &\geq 0 \end{aligned}$$

Step 3 : ***Determine which ranges correspond to the inequality***
We need to figure out which values of x satisfy the inequality. From the answers we have five regions to consider.



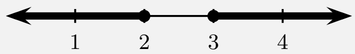
Step 4 : ***Determine whether the function is negative or positive in each of the regions***
Let $f(x) = x^2 - 5x + 6$. For each region, choose any point in the region and evaluate the function.

		$f(x)$	sign of $f(x)$
Region A	$x < 2$	$f(1) = 2$	+
Region B	$x = 2$	$f(2) = 0$	+
Region C	$2 < x < 3$	$f(2{,}5) = -2{,}5$	-
Region D	$x = 3$	$f(3) = 0$	+
Region E	$x > 3$	$f(4) = 2$	+

We see that the function is positive for $x \leq 2$ and $x \geq 3$.

Step 5 : ***Write the final answer and represent on a number line***

We see that $x^2 - 5x + 6 \geq 0$ is true for $x \leq 2$ and $x \geq 3$.



Example 3: *Solving Quadratic Inequalities*

QUESTION

Solve the quadratic inequality $-x^2 - 3x + 5 > 0$.

SOLUTION

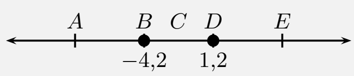
Step 1 : ***Determine how to approach the problem***

Let $f(x) = -x^2 - 3x + 5$. $f(x)$ cannot be factorised so, use the quadratic formula to determine the roots of $f(x)$. The x-intercepts are solutions to the quadratic equation

$$\begin{aligned} -x^2 - 3x + 5 &= 0 \\ x^2 + 3x - 5 &= 0 \\ \therefore x &= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-5)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{29}}{2} \\ x_1 &= \frac{-3 - \sqrt{29}}{2} = -4{,}2 \\ x_2 &= \frac{-3 + \sqrt{29}}{2} = 1{,}2 \end{aligned}$$

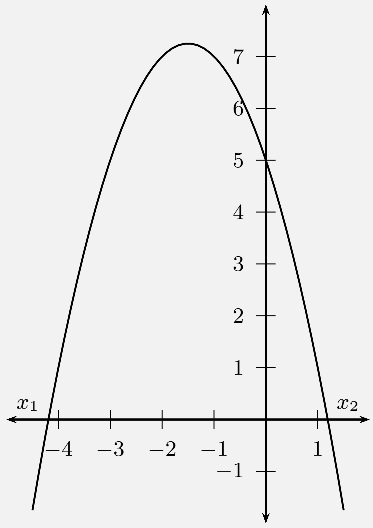
Step 2 : ***Determine which ranges correspond to the inequality***

We need to figure out which values of x satisfy the inequality. From the answers we have five regions to consider.



Step 3 : ***Determine whether the function is negative or positive in each of the regions***

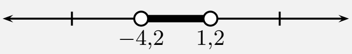
We can use another method to determine the sign of the function over different regions, by drawing a rough sketch of the graph of the function. We know that the roots of the function correspond to the x-intercepts of the graph. Let $g(x) = -x^2 - 3x + 5$. We can see that this is a parabola with a maximum turning point that intersects the x-axis at $-4{,}2$ and $1{,}2$.



It is clear that $g(x) > 0$ for $x_1 < x < x_2$

Step 4 : ***Write the final answer and represent the solution graphically***

$-x^2 - 3x + 5 > 0$ for $-4{,}2 < x < 1{,}2$



When working with an inequality in which the variable is in the denominator, a different approach is needed.

Example 4: *Non-linear inequality with the variable in the denominator*

QUESTION

Solve $\dfrac{2}{x+3} \leq \dfrac{1}{x-3}$

SOLUTION

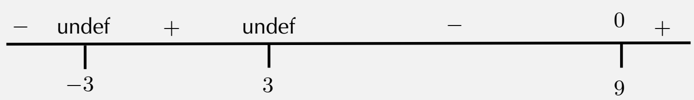
Step 1 : ***Subtract $\frac{1}{x-3}$ from both sides***

$$\frac{2}{x+3} - \frac{1}{x-3} \leq 0$$

Step 2 : ***Simplify the fraction by finding LCD***

$$\frac{2(x-3)-(x+3)}{(x+3)(x-3)} \leq 0$$
$$\frac{x-9}{(x+3)(x-3)} \leq 0$$

Step 3 : ***Draw a number line for the inequality***



We see that the expression is negative for $x < -3$ or $3 < x \leq 9$.

Step 4 : ***Write the final answer***

$$x < -3 \quad or \quad 3 < x \leq 9$$

See video: VMfeu at www.everythingmaths.co.za

Chapter 8 End of Chapter Exercises

Solve the following inequalities and show your answer on a number line:

1. Solve: $x^2 - x < 12$.
2. Solve: $3x^2 > -x + 4$
3. Solve: $y^2 < -y - 2$
4. Solve: $-t^2 + 2t > -3$
5. Solve: $s^2 - 4s > -6$
6. Solve: $0 \geq 7x^2 - x + 8$
7. Solve: $0 \geq -4x^2 - x$
8. Solve: $0 \geq 6x^2$
9. Solve: $2x^2 + x + 6 \leq 0$
10. Solve for x if: $\frac{x}{x-3} < 2$ and $x \neq 3$.
11. Solve for x if: $\frac{4}{x-3} \leq 1$.

12. Solve for x if: $\frac{4}{(x-3)^2} < 1$.
13. Solve for x: $\frac{2x-2}{x-3} > 3$
14. Solve for x: $\frac{-3}{(x-3)(x+1)} < 0$
15. Solve: $(2x-3)^2 < 4$
16. Solve: $2x \leq \frac{15-x}{x}$
17. Solve for x: $\frac{x^2+3}{3x-2} \leq 0$
18. Solve: $x - 2 \geq \frac{3}{x}$
19. Solve for x: $\frac{x^2+3x-4}{5+x^4} \leq 0$
20. Determine all real solutions: $\frac{x-2}{3-x} \geq 1$

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(1.) 019t	(2.) 019u	(3.) 019v	(4.) 019w	(5.) 019x	(6.) 019y
(7.) 019z	(8.) 01a0	(9.) 01a1	(10.) 01a2	(11.) 01a3	(12.) 01a4
(13.) 01a5	(14.) 01a6	(15.) 01a7	(16.) 01a8	(17.) 01a9	(18.) 01aa
(19.) 01ab	(20.) 01ac				

Solving Simultaneous Equations 9

9.1 Introduction

EMBAO

In Grade 10, you learnt how to solve sets of simultaneous equations where both equations were linear (i.e. had the highest power equal to 1). In this chapter, you will learn how to solve sets of simultaneous equations where one is linear and one is quadratic. As in Grade 10, the solution will be found both algebraically and graphically.

The only difference between a system of linear simultaneous equations and a system of simultaneous equations with one linear and one quadratic equation, is that the second system will have at most two solutions.

An example of a system of simultaneous equations with one linear equation and one quadratic equation is:

$$\begin{aligned} y - 2x &= -4 \\ x^2 + y &= 4 \end{aligned} \tag{9.1}$$

See introductory video: VMfwh at www.everythingmaths.co.za

9.2 Graphical Solution

EMBAP

The method of graphically finding the solution to one linear and one quadratic equation is identical to systems of linear simultaneous equations.

Method: Graphical solution to a system of simultaneous equations with one linear and one quadratic equation

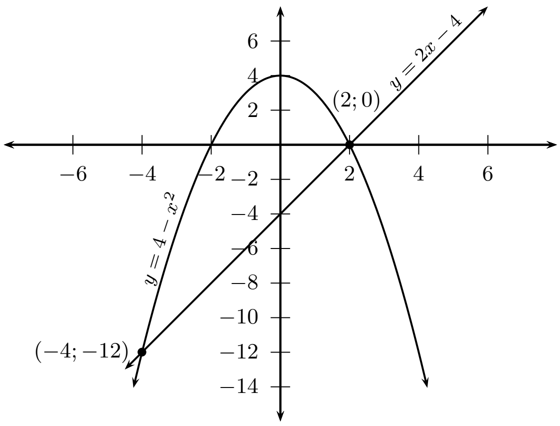
EMBAQ

1. Make y the subject of each equation.
2. Draw the graphs of each equation as defined above.
3. The solution of the set of simultaneous equations is given by the intersection points of the two graphs.

For this example, making y the subject of each equation, gives:

$$\begin{aligned} y &= 2x - 4 \\ y &= 4 - x^2 \end{aligned}$$

Plotting the graph of each equation, gives a straight line for the first equation and a parabola for the second equation.



The parabola and the straight line intersect at two points: $(2; 0)$ and $(-4; -12)$. Therefore, the solutions to the system of equations in (9.1) is $x = 2; y = 0$ and $x = -4; y = 12$

Example 1: *Simultaneous Equations*

QUESTION

Solve graphically:

$$\begin{aligned} y - x^2 + 9 &= 0 \\ y + 3x - 9 &= 0 \end{aligned}$$

SOLUTION

Step 1 : ***Make y the subject of the equation***

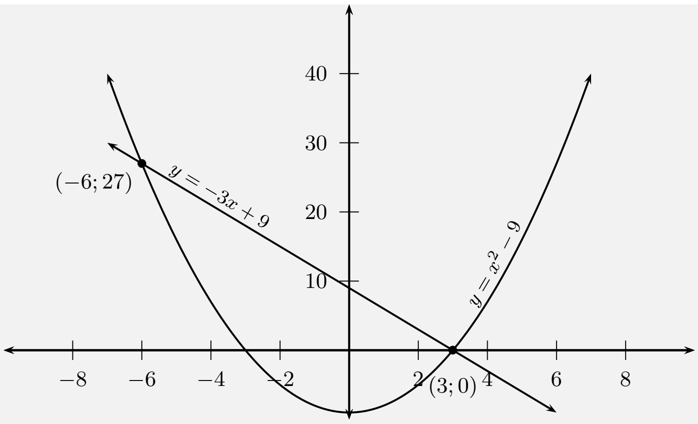
For the first equation:

$$\begin{aligned} y - x^2 + 9 &= 0 \\ y &= x^2 - 9 \end{aligned}$$

and for the second equation:

$$\begin{aligned} y + 3x - 9 &= 0 \\ y &= -3x + 9 \end{aligned}$$

Step 2 : ***Draw the graphs corresponding to each equation.***



Step 3 : ***Find the intersection of the graphs.***
The graphs intersect at $(-6; 27)$ and at $(3; 0)$.

Step 4 : ***Write the solution of the system of simultaneous equations as given by the intersection of the graphs.***
The first solution is $x = -6$ and $y = 27$. The second solution is $x = 3$ and $y = 0$.

Exercise 9 - 1

Solve the following systems of equations graphically. Leave your answer in surd form, where appropriate.

1. $b^2 - 1 - a = 0; a + b - 5 = 0$
2. $x + y - 10 = 0; x^2 - 2 - y = 0$
3. $6 - 4x - y = 0; 12 - 2x^2 - y = 0$
4. $x + 2y - 14 = 0; x^2 + 2 - y = 0$
5. $2x + 1 - y = 0; 25 - 3x - x^2 - y = 0$

More practice video solutions or help at www.everythingmaths.co.za

(1.) 01ad (2.) 01ae (3.) 01af (4.) 01ag (5.) 01ah

9.3 Algebraic Solution

EMBAR

The algebraic method of solving simultaneous equations is by substitution.

For example the solution of

$$y - 2x = -4 \tag{9.2}$$
$$x^2 + y = 4 \tag{9.3}$$

is:

In (9.2)

$$y = 2x - 4 \tag{9.4}$$

Substitute (9.4) into (9.3):

$$x^2 + (2x - 4) = 4$$
$$x^2 + 2x - 8 = 0$$

Factorise to get:

$$(x + 4)(x - 2) = 0$$
$$\therefore x = -4 \quad \text{and} \quad x = 2$$

Substitute the values of x into (9.4) to find y:

$y = 2(-4) - 4$	$y = 2(2) - 4$
$y = -12$	$y = 0$
$(-4; -12)$	$(2; 0)$

As expected, these solutions are identical to those obtained by the graphical solution.

Example 2: *Simultaneous Equations*

QUESTION

Solve algebraically:

$$y - x^2 + 9 = 0 \tag{9.5}$$
$$y + 3x - 9 = 0 \tag{9.6}$$

SOLUTION

Step 1 : ***Make y the subject of the linear equation***
In (9.5):

$$y + 3x - 9 = 0$$
$$y = -3x + 9 \tag{9.7}$$

Step 2 : ***Substitute into the quadratic equation***

Substitute (9.7) into (9.5):

$$\begin{aligned} (-3x+9)-x^2+9 &= 0 \\ x^2+3x-18 &= 0 \end{aligned}$$

Factorise to get:

$$\begin{aligned} (x+6)(x-3) &= 0 \\ \therefore x=-6 \quad &\text{and} \quad x=3 \end{aligned}$$

Step 3 : ***Substitute the values for x into the first equation to calculate the corresponding y-values.***

Substitute x into 9.5:

$$\begin{aligned} y &= -3(-6)+9 \\ &= 27 \\ &(-6;27) \end{aligned} \qquad \begin{aligned} y &= -3(3)+9 \\ &= 0 \\ &(3;0) \end{aligned}$$

Step 4 : ***Write the solution of the system of simultaneous equations.***

The first solution is $x=-6$ and $y=27$. The second solution is $x=3$ and $y=0$.

Chapter 9 End of Chapter Exercises

Solve the following systems of equations algebraically. Leave your answer in surd form, where appropriate.

1. $a+b=5$; $a-b^2+3b-5=0$
2. $a-b+1=0$; $a-b^2+5b-6=0$
3. $a-\frac{(2b+2)}{4}=0$ $a-2b^2+3b+5=0$
4. $a+2b-4=0$; $a-2b^2-5b+3=0$
5. $a-2+3b=0$; $a-9+b^2=0$
6. $a-b-5=0$; $a-b^2=0$
7. $a-b-4=0$; $a+2b^2-12=0$
8. $a+b-9=0$; $a+b^2-18=0$
9. $a-3b+5=0$; $a+b^2-4b=0$
10. $a+b-5=0$; $a-b^2+1=0$
11. $a-2b-3=0$; $a-3b^2+4=0$
12. $a-2b=0$; $a-b^2-2b+3=0$
13. $a-3b=0$; $a-b^2+4=0$
14. $a-2b-10=0$; $a-b^2-5b=0$
15. $a-3b-1=0$; $a-2b^2-b+3=0$
16. $a-3b+1=0$; $a-b^2=0$
17. $a+6b-5=0$; $a-b^2-8=0$

18. $a - 2b + 1 = 0$; $a - 2b^2 - 12b + 4 = 0$

19. $2a + b - 2 = 0$; $8a + b^2 - 8 = 0$

20. $a + 4b - 19 = 0$; $8a + 5b^2 - 101 = 0$

21. $a + 4b - 18 = 0$; $2a + 5b^2 - 57 = 0$

More practice video solutions or help at www.everythingmaths.co.za

(1.) 020u	(2.) 020v	(3.) 020w	(4.) 020x	(5.) 020y	(6.) 020z
(7.) 0210	(8.) 0211	(9.) 0212	(10.) 0213	(11.) 0214	(12.) 0215
(13.) 0216	(14.) 0217	(15.) 0218	(16.) 0219	(17.) 021a	(18.) 021b
(19.) 021c	(20.) 021d	(21.) 021e			

Mathematical Models 10

10.1 Introduction

EMBAS

Up until now, you have only learnt how to solve equations and inequalities, but there has not been much application of what you have learnt. This chapter builds on this knowledge and introduces you to the idea of a *mathematical model*, which uses mathematical concepts to solve real-world problems.

See introductory video: VMfjh at www.everythingmaths.co.za

10.2 Mathematical Models

EMBAT

> *DEFINITION: Mathematical Model*
>
> A mathematical model is a method of using the mathematical language to describe the behaviour of a physical system. Mathematical models are used particularly in the natural sciences and engineering disciplines (such as physics, biology, and electrical engineering) but also in the social sciences (such as economics, sociology and political science); physicists, engineers, computer scientists, and economists use mathematical models most extensively.

A mathematical model is an equation (or a set of equations for the more difficult problems) that describes a particular situation. For example, if Anna receives R3 for each time she helps her mother wash the dishes and R5 for each time she helps her father cut the grass, how much money will Anna earn if she helps her mother to wash the dishes five times and helps her father to wash the car twice? The first step to modelling is to write the equation, that describes the situation. To calculate how much Anna will earn we see that she will earn :

$$\begin{array}{llll} & 5 & \times \text{R}3 & \text{for washing the dishes} \\ + & 2 & \times \text{R}5 & \text{for cutting the grass} \\ = & \text{R}15 & + \text{R}10 & \\ = & \text{R}25 & & \end{array}$$

If however, we ask: "What is the equation if Anna helps her mother x times and her father y times?", then we have:

$$\text{Total earned} = (x \times \text{R}3) + (y \times \text{R}5)$$

10.3 Real-World Applications

EMBAU

Some examples of where mathematical models are used in the real-world are:

1. To model population growth
2. To model effects of air pollution
3. To model effects of global warming
4. In computer games
5. In the sciences (e.g. physics, chemistry, biology) to understand how the natural world works
6. In simulators that are used to train people in certain jobs, like pilots, doctors and soldiers
7. In medicine to track the progress of a disease

Activity: *Simple Models*

In order to get used to the idea of mathematical models, try the following simple models. Write an equation that describes the following real-world situations, mathematically:

1. Jack and Jill both have colds. Jack sneezes twice for each sneeze of Jill's. If Jill sneezes x times, write an equation describing how many times they both sneezed?
2. It rains half as much in July as it does in December. If it rains y mm in July, write an expression relating the rainfall in July and December.
3. Zane can paint a room in 4 hours. Billy can paint a room in 2 hours. How long will it take both of them to paint a room together?
4. 25 years ago, Arthur was 5 more than $\frac{1}{3}$ as old as Lee was. Today, Lee is 26 less than twice Arthur's age. How old is Lee?
5. Kevin has played a few games of ten-pin bowling. In the third game, Kevin scored 80 more than in the second game. In the first game Kevin scored 110 less than the third game. His total score for the first two games was 208. If he wants an average score of 146, what must he score on the fourth game?
6. Erica has decided to treat her friends to coffee at the Corner Coffee House. Erica paid R$54{,}00$ for four cups of cappuccino and three cups of filter coffee. If a cup of cappuccino costs R$3{,}00$ more than a cup of filter coffee, calculate how much each type of coffee costs?
7. The product of two integers is 95. Find the integers if their total is 24.

Example 1: *Mathematical Modelling of Falling Objects*

QUESTION

When an object is dropped or thrown downward, the distance, d, that it falls in time, t is described by the following equation:

$$s = 5t^2 + v_0t$$

In this equation, v_0 is the initial velocity, in $\mathrm{m\cdot s^{-1}}$. Distance is measured in meters and time is measured in seconds. Use the equation to find how far an object will fall in $2\ \mathrm{s}$ if it is thrown downward at an initial velocity of $10\ \mathrm{m\cdot s^{-1}}$.

SOLUTION

Step 1 : ***Identify what is given for each problem***
We are given an expression to calculate distance travelled by a falling object in terms of initial velocity and time. We are also given the initial velocity and time and are required to calculate the distance travelled.

Step 2 : ***List all known and unknown information***

- $v_0 = 10\ \mathrm{m\cdot s^{-1}}$
- $t = 2\ \mathrm{s}$
- $s = ?\ \mathrm{m}$

Step 3 : ***Substitute values into expression***

$$\begin{aligned} s &= 5t^2 + v_0t \\ &= 5(2)^2 + (10)(2) \\ &= 5(4) + 20 \\ &= 20 + 20 \\ &= 40 \end{aligned}$$

Step 4 : ***Write the final answer***
The object will fall $40\ \mathrm{m}$ in $2\ \mathrm{s}$ if it is thrown downward at an initial velocity of $10\ \mathrm{m\cdot s^{-1}}$.

Example 2: *Another Mathematical Modelling of Falling Objects*

QUESTION

When an object is dropped or thrown downward, the distance, d, that it falls in time, t is described by the following equation:

$$s = 5t^2 + v_0 t$$

In this equation, v_0 is the initial velocity, in $\mathrm{m \cdot s^{-1}}$. Distance is measured in meters and time is measured in seconds. Use the equation find how long it takes for the object to reach the ground if it is dropped from a height of $2000\,\mathrm{m}$. The initial velocity is $0\ \mathrm{m \cdot s^{-1}}$.

SOLUTION

Step 1 : ***Identify what is given for each problem***

We are given an expression to calculate distance travelled by a falling object in terms of initial velocity and time. We are also given the initial velocity and distance travelled and are required to calculate the time it takes the object to travel the distance.

Step 2 : ***List all known and unknown information***

- $v_0 = 0\,\mathrm{m \cdot s^{-1}}$
- $t = ?\,\mathrm{s}$
- $s = 2000\ \mathrm{m}$

Step 3 : ***Substitute values into expression***

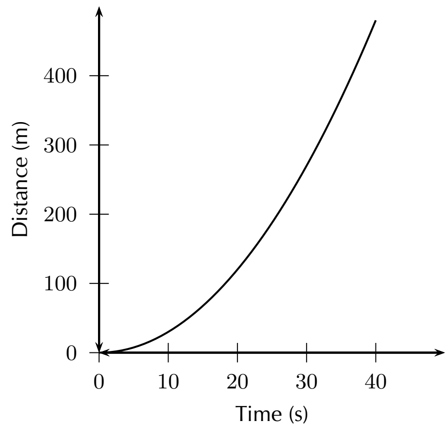
$$\begin{aligned} s &= 5t^2 + v_0 t \\ 2000 &= 5t^2 + (0)(2) \\ 2000 &= 5t^2 \\ t^2 &= \frac{2000}{5} \\ &= 400 \\ \therefore \quad t &= 20\ \mathrm{s} \end{aligned}$$

Step 4 : ***Write the final answer***

The object will take $20\ \mathrm{s}$ to reach the ground if it is dropped from a height of $2\,000\ \mathrm{m}$.

Activity: *Mathematical Modelling*

The graph below shows how the distance travelled by a car depends on time. Use the graph to answer the following questions.



1. How far does the car travel in $20\ \text{s}$?
2. How long does it take the car to travel $300\ \text{m}$?

Example 3: *More Mathematical Modelling*

QUESTION

A researcher is investigating the number of trees in a forest over a period of n years. After investigating numerous data, the following data model emerged:

Year	*Number of trees (in hundreds)*
1	1
2	3
3	9
4	27

1. *How many trees, in hundreds, are there in the sixth year if this pattern is continued?*
2. *Determine an algebraic expression that describes the number of trees in the n^{th} year in the forest.*
3. *Do you think this model, which determines the number of trees in the forest, will continue indefinitely? Give a reason for your answer.*

SOLUTION

Step 1 : ***Find the pattern***

The pattern is $3^0; 3^1; 3^2; 3^3; \ldots$

Therefore, three to the power one less than the year.

Step 2 : ***Trees in year 6***

$$\text{Year } 6 : 3^5 \text{ hundred } = 243 \text{ hundred } = 24300$$

Step 3 : ***Algebraic expression for year n***

$$\text{Number of trees } = 3^{n-1} \text{ hundred}$$

Step 4 : ***Conclusion***
No, the number of trees will not increase indefinitely. The number of trees will increase for some time. Yet, eventually the number of trees will not increase any more. It will be limited by factors such as the limited amount of water and nutrients available in the ecosystem.

Example 4: *Setting up an equation*

QUESTION

Currently the subscription to a gym for a single member is R1 000 annually while family membership is R1 500. The gym is considering raising all memberships fees by the same amount. If this is done then the single membership will cost $\frac{5}{7}$ of the family membership. Determine the proposed increase.

SOLUTION

Step 1 : ***Summarise the information in a table***
Let the proposed increase be x.

	Now	After increase
Single	1 000	$1\ 000 + x$
Family	1 500	$1\ 500 + x$

Step 2 : ***Set up an equation***

$$1\ 000 + x = \frac{5}{7}(1\ 500 + x)$$

Step 3 : ***Solve the equation.***

$$\begin{aligned} 7\ 000 + 7x &= 7\ 500 + 5x \\ 2x &= 500 \\ x &= 250 \end{aligned}$$

Step 4 : ***Write down the answer***
Therefore the increase is R250.

Extension: *Simulations*

A simulation is an attempt to model a real-life situation on a computer so that it can be studied to see how the system works. By changing variables, predictions may be made about the behaviour of the system. Simulation is used in many contexts, including the modelling of natural systems or human systems in order to gain insight into their functioning. Other contexts include simulation of technology for performance optimisation, safety engineering, testing, training and education. Simulation can be used to show the eventual real effects of alternative conditions and courses of action.

Simulation in education Simulations in education are somewhat like training simulations. They focus on specific tasks. In the past, video has been used for teachers and education students to observe, problem solve and role play; however, a more recent use of simulations in education is that of animated narrative vignettes (ANV). ANVs are cartoon-like video narratives of hypothetical and reality-based stories involving classroom teaching and learning. ANVs have been used to assess knowledge, problem solving skills and dispositions of children and pre-service and in-service teachers.

Chapter 10 End of Chapter Exercises

1. When an object is dropped or thrown downward, the distance, d, that it falls in time, t, is described by the following equation:

   $$s = 5t^2 + v_0 t$$

   In this equation, v_0 is the initial velocity, in $\mathrm{m \cdot s^{-1}}$. Distance is measured in meters and time is measured in seconds. Use the equation to find how long it takes a tennis ball to reach the ground if it is thrown downward from a hot-air balloon that is $500\ \mathrm{m}$ high. The tennis ball is thrown at an initial velocity of $5\ \mathrm{m \cdot s^{-1}}$.

2. The table below lists the times that Sheila takes to walk the given distances.

Time (minutes)	5	10	15	20	25	30
Distance (km)	1	2	3	4	5	6

   Plot the points.

   If the relationship between the distances and times is linear, find the equation of the straight line, using any two points. Then use the equation to answer the following questions:

   (a) How long will it take Sheila to walk 21 km?

   (b) How far will Sheila walk in 7 minutes?

   If Sheila were to walk half as fast as she is currently walking, what would the graph of her distances and times look like?

3. The power P (in watts) supplied to a circuit by a 12 volt battery is given by the formula $P = 12I - 0{,}5I^2$ where I is the current in amperes.

   (a) Since both power and current must be greater than 0, find the limits of the current that can be drawn by the circuit.

   (b) Draw a graph of $P = 12I - 0{,}5I^2$ and use your answer to the first question, to define the extent of the graph.

   (c) What is the maximum current that can be drawn?

(d) From your graph, read off how much power is supplied to the circuit when the current is 10 Amperes. Use the equation to confirm your answer.

(e) At what value of current will the power supplied be a maximum?

4. You are in the lobby of a business building waiting for the lift. You are late for a meeting and wonder if it will be quicker to take the stairs. There is a fascinating relationship between the number of floors in the building, the number of people in the lift and how often it will stop:

   If N people get into a lift at the lobby and the number of floors in the building is F, then the lift can be expected to stop

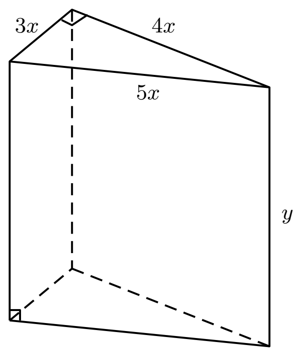
   $$F - F\left(\frac{F-1}{F}\right)^N$$

   times.

   (a) If the building has 16 floors and there are 9 people who get into the lift, how many times is the lift expected to stop?

   (b) How many people would you expect in a lift, if it stopped 12 times and there are 17 floors?

5. A wooden block is made as shown in the diagram. The ends are right-angled triangles having sides $3x$, $4x$ and $5x$. The length of the block is y. The total surface area of the block is 3600 cm^2.



   Show that

   $$y = \frac{300 - x^2}{x}$$

6. A stone is thrown vertically upwards and its height (in metres) above the ground at time t (in seconds) is given by:

   $$h(t) = 35 - 5t^2 + 30t$$

   Find its initial height above the ground.

7. After doing some research, a transport company has determined that the rate at which petrol is consumed by one of its large carriers, travelling at an average speed of x km per hour, is given by:

   $$P(x) = \frac{55}{2x} + \frac{x}{200} \quad \text{litres per kilometre}$$

   Assume that the petrol costs R$4{,}00$ per litre and the driver earns R$18{,}00$ per hour (travelling time). Now deduce that the total cost, C, in Rands, for a 2 000 km trip is given by:

$$C(x) = \frac{256000}{x} + 40x$$

8. During an experiment the temperature T (in degrees Celsius), varies with time t (in hours), according to the formula:

$$T(t) = 30 + 4t - \frac{1}{2}t^2 \quad t \in [1; 10]$$

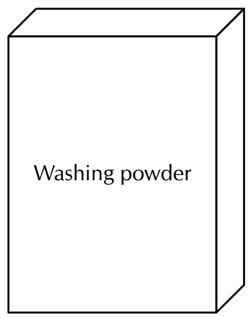
   (a) Determine an expression for the rate of change of temperature with time.
   (b) During which time interval was the temperature dropping?

9. In order to reduce the temperature in a room from $28^\circ\ C$, a cooling system is allowed to operate for 10 minutes. The room temperature, T after t minutes is given in $^\circ\ C$ by the formula:

$$T = 28 - 0{,}008t^3 - 0{,}16t \quad \text{where } t \in [0; 10]$$

   (a) At what rate (rounded off to two decimal places) is the temperature falling when $t = 4$ minutes?
   (b) Find the lowest room temperature reached during the 10 minutes for which the cooling system operates, by drawing a graph.

10. A washing powder box has the shape of a rectangular prism as shown in the diagram below. The box has a volume of 480 cm^3, a breadth of 4 cm and a length of x cm.



Show that the total surface area of the box (in cm^2) is given by:

$$A = 8x + 960x^{-1} + 240$$

More practice video solutions or help at www.everythingmaths.co.za

(1.) 010w (2.) 010x (3.) 010y (4.) 010z (5.) 0110 (6.) 0111
(7.) 0112 (8.) 0113 (9.) 0114 (10.) 0115

Quadratic Functions and Graphs 11

11.1 Introduction

EMBAV

In Grade 10, you studied graphs of many different forms. Here you will learn how to sketch and interpret more general quadratic functions.

See introductory video: VMfkg at www.everythingmaths.co.za

11.2 Functions of the Form $y = a(x + p)^2 + q$

EMBAW

This form of the quadratic function is slightly more complex than the form studied in Grade 10, $y = ax^2 + q$. The general shape and position of the graph of the function of the form $f(x) = a(x + p)^2 + q$ is shown in Table 11.1.

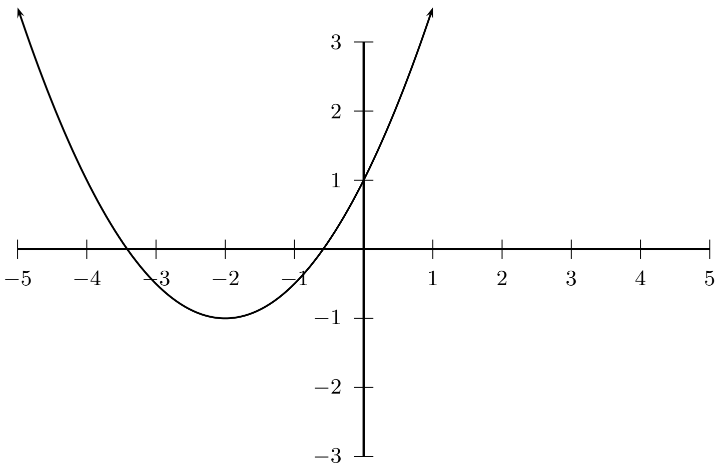


Figure 11.1: Graph of $f(x) = \frac{1}{2}(x + 2)^2 - 1$

Activity:	*Functions of the Form* $y = a(x + p)^2 + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = (x - 2)^2$
   (b) $b(x) = (x - 1)^2$
   (c) $c(x) = x^2$

(d) $d(x) = (x+1)^2$

(e) $e(x) = (x+2)^2$

Use your results to deduce the effect of p.

2. On the same set of axes, plot the following graphs:

   (a) $f(x) = (x-2)^2 + 1$

   (b) $g(x) = (x-1)^2 + 1$

   (c) $h(x) = x^2 + 1$

   (d) $j(x) = (x+1)^2 + 1$

   (e) $k(x) = (x+2)^2 + 1$

   Use your results to deduce the effect of q.

3. Following the general method of the above activities, choose your own values of p and q to plot 5 different graphs (on the same set of axes) of $y = a(x+p)^2 + q$ to deduce the effect of a.

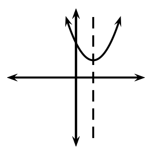
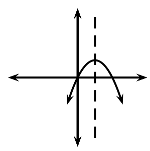
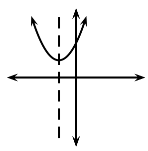
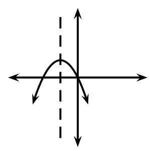
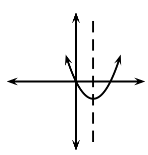
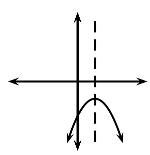
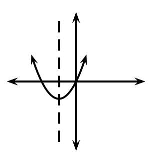
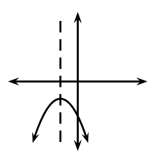
From your graphs, you should have found that a affects whether the graph makes a smile or a frown. If $a < 0$, the graph makes a frown and if $a > 0$ then the graph makes a smile. This was shown in Grade 10.

You should have also found that the value of q affects whether the turning point of the graph is above the x-axis ($q < 0$) or below the x-axis ($q > 0$).

You should have also found that the value of p affects whether the turning point is to the left of the y-axis ($p > 0$) or to the right of the y-axis ($p < 0$).

These different properties are summarised in Table 11.1. The axes of symmetry for each graph is shown as a dashed line.

Table 11.1: Table summarising general shapes and positions of functions of the form $y = a(x+p)^2 + q$. The axes of symmetry are shown as dashed lines.

	$p < 0$		$p > 0$	
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q \geq 0$				
$q \leq 0$				

See simulation: VMflm at www.everythingmaths.co.za)

Domain and Range

EMBAX

For $f(x) = a(x+p)^2 + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = a(x+p)^2 + q$ depends on whether the value for a is positive or negative. We will consider these two cases separately.

If $a > 0$ then we have:

$$\begin{aligned} (x+p)^2 &\geq 0 \quad \text{(The square of an expression is always positive)} \\ a(x+p)^2 &\geq 0 \quad \text{(Multiplication by a positive number maintains the nature of the inequality)} \\ a(x+p)^2 + q &\geq q \\ f(x) &\geq q \end{aligned}$$

This tells us that for all values of x, $f(x)$ is always greater than or equal to q. Therefore if $a > 0$, the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$.

Similarly, it can be shown that if $a < 0$ that the range of $f(x) = a(x+p)^2+q$ is $\{f(x) : f(x) \in (-\infty,q]\}$. This is left as an exercise.

For example, the domain of $g(x) = (x-1)^2 + 2$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ can be calculated as follows:

$$\begin{aligned} (x-p)^2 &\geq 0 \\ (x+p)^2 + 2 &\geq 2 \\ g(x) &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

Exercise 11 - 1

1. Given the function $f(x) = (x-4)^2 - 1$. Give the range of $f(x)$.
2. What is the domain of the equation $y = 2x^2 - 5x - 18$?

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(1.) 0116 (2.) 0117

Intercepts

EMBAY

For functions of the form, $y = a(x+p)^2 + q$, the details of calculating the intercepts with the x and y axes is given.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= a(x+p)^2 + q && (11.1) \\ y_{int} &= a(0+p)^2 + q && (11.2) \\ &= ap^2 + q && (11.3) \end{aligned}$$

If $p = 0$, then $y_{int} = q$.

For example, the y-intercept of $g(x) = (x-1)^2 + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} g(x) &= (x-1)^2 + 2 \\ y_{int} &= (0-1)^2 + 2 \\ &= (-1)^2 + 2 \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

The x-intercepts are calculated as follows:

$$y = a(x+p)^2 + q \quad (11.4)$$

$$0 = a(x_{int}+p)^2 + q \quad (11.5)$$

$$a(x_{int}+p)^2 = -q \quad (11.6)$$

$$x_{int} + p = \pm\sqrt{-\frac{q}{a}} \quad (11.7)$$

$$x_{int} = \pm\sqrt{-\frac{q}{a}} - p \quad (11.8)$$

However, (11.8) is only valid if $-\frac{q}{a} > 0$ which means that either $q < 0$ or $a < 0$ but not both. This is consistent with what we expect, since if $q > 0$ and $a > 0$ then $-\frac{q}{a}$ is negative and in this case the graph lies above the x-axis and therefore does not intersect the x-axis. If however, $q > 0$ and $a < 0$, then $-\frac{q}{a}$ is positive and the graph is hat shaped with turning point above the x-axis and should have two x-intercepts. Similarly, if $q < 0$ and $a > 0$ then $-\frac{q}{a}$ is also positive, and the graph should intersect with the x-axis twice.

For example, the x-intercepts of $g(x) = (x-1)^2 + 2$ are given by setting $y = 0$ to get:

$$\begin{aligned} g(x) &= (x-1)^2 + 2 \\ 0 &= (x_{int}-1)^2 + 2 \\ -2 &= (x_{int}-1)^2 \end{aligned}$$

which has no real solutions. Therefore, the graph of $g(x) = (x-1)^2 + 2$ does not have any x-intercepts.

Exercise 11 - 2

1. Find the x- and y-intercepts of the function $f(x) = (x-4)^2 - 1$.
2. Find the intercepts with both axes of the graph of $f(x) = x^2 - 6x + 8$.
3. Given: $f(x) = -x^2 + 4x - 3$. Calculate the x- and y-intercepts of the graph of f.

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(1.) 0118 (2.) 0119 (3.) 011a

Turning Points

EMBAZ

The turning point of the function of the form $f(x) = a(x+p)^2 + q$ is given by examining the range of the function. We know that if $a > 0$ then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$ and if $a < 0$ then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.

So, if $a > 0$, then the lowest value that $f(x)$ can take on is q. Solving for the value of x at which $f(x) = q$ gives:

$$\begin{aligned} q &= a(x+p)^2 + q \\ 0 &= a(x+p)^2 \\ 0 &= (x+p)^2 \\ 0 &= x+p \\ x &= -p \end{aligned}$$

$\therefore x = -p$ at $f(x) = q$. The co-ordinates of the (minimal) turning point are therefore $(-p; q)$.

Similarly, if $a < 0$, then the highest value that $f(x)$ can take on is q and the co-ordinates of the (maximal) turning point are $(-p; q)$.

Exercise 11 - 3

1. Determine the turning point of the graph of $f(x) = x^2 - 6x + 8$.
2. Given: $f(x) = -x^2 + 4x - 3$. Calculate the co-ordinates of the turning point of f.
3. Find the turning point of the following function:
   $y = \frac{1}{2}(x+2)^2 - 1$.

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(1.) 011b (2.) 011c (3.) 011d

Axes of Symmetry

EMBBA

There is only one axis of symmetry for the function of the form $f(x) = a(x+p)^2 + q$. This axis passes through the turning point and is parallel to the y-axis. Since the x-coordinate of the turning point is $x = -p$, this is the axis of symmetry.

Exercise 11 - 4

1. Find the equation of the axis of symmetry of the graph $y = 2x^2 - 5x - 18$.
2. Write down the equation of the axis of symmetry of the graph of
   $y = 3(x-2)^2 + 1$.
3. Write down an example of an equation of a parabola where the y-axis is the axis of symmetry.

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(1.) 011e (2.) 011f (3.) 011g

Sketching Graphs of the Form $f(x) = a(x + p)^2 + q$ EMBBB

In order to sketch graphs of the form $f(x) = a(x+p)^2 + q$, we need to determine five characteristics:

1. sign of a
2. domain and range
3. turning point
4. y-intercept
5. x-intercept (if appropriate)

For example, sketch the graph of $g(x) = -\frac{1}{2}(x+1)^2 - 3$. Mark the intercepts, turning point and axis of symmetry.

Firstly, we determine that $a < 0$. This means that the graph will have a maximal turning point.

The domain of the graph is $\{x : x \in \mathbb{R}\}$ because $f(x)$ is defined for all $x \in \mathbb{R}$. The range of the graph is determined as follows:

$$\begin{aligned}
(x+1)^2 &\geq 0 \\
-\frac{1}{2}(x+1)^2 &\leq 0 \\
-\frac{1}{2}(x+1)^2 - 3 &\leq -3 \\
\therefore f(x) &\leq -3
\end{aligned}$$

Therefore the range of the graph is $\{f(x) : f(x) \in (-\infty, -3]\}$.

Using the fact that the maximum value that $f(x)$ achieves is -3, then the y-coordinate of the turning point is -3. The x-coordinate is determined as follows:

$$\begin{aligned}
-\frac{1}{2}(x+1)^2 - 3 &= -3 \\
-\frac{1}{2}(x+1)^2 - 3 + 3 &= 0 \\
-\frac{1}{2}(x+1)^2 &= 0 \\
\text{Divide both sides by } -\tfrac{1}{2}: \quad (x+1)^2 &= 0 \\
\text{Take square root of both sides:} \quad x+1 &= 0 \\
\therefore \quad x &= -1
\end{aligned}$$

The coordinates of the turning point are: $(-1; -3)$.

The y-intercept is obtained by setting $x = 0$. This gives:

$$\begin{aligned}
y_{int} &= -\frac{1}{2}(0+1)^2 - 3 \\
&= -\frac{1}{2}(1) - 3 \\
&= -\frac{1}{2} - 3 \\
&= -\frac{1}{2} - 3 \\
&= -\frac{7}{2}
\end{aligned}$$

The x-intercept is obtained by setting $y = 0$. This gives:

$$\begin{aligned} 0 &= -\frac{1}{2}(x_{int}+1)^2 - 3 \\ 3 &= -\frac{1}{2}(x_{int}+1)^2 \\ -3\,.\,2 &= (x_{int}+1)^2 \\ -6 &= (x_{int}+1)^2 \end{aligned}$$

which has no real solutions. Therefore, there are no x-intercepts.

We also know that the axis of symmetry is parallel to the y-axis and passes through the turning point.

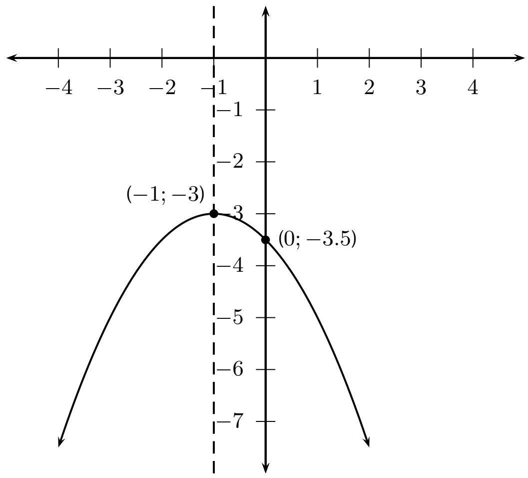

Figure 11.2: Graph of the function $f(x) = -\frac{1}{2}(x+1)^2 - 3$

See video: VMfkk at www.everythingmaths.co.za

Exercise 11 - 5

1. Draw the graph of $y = 3(x-2)^2 + 1$ showing all the intercepts with the axes as well as the coordinates of the turning point.

2. Draw a neat sketch graph of the function defined by $y = ax^2 + bx + c$ if $a > 0$; $b < 0$; $b^2 = 4ac$.

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(1.) 011h (2.) 011i

Writing an Equation of a Shifted Parabola

EMBBC

Given a parabola with equation $y = x^2 - 2x - 3$. The graph of the parabola is shifted one unit to the right. Or else the y-axis shifts one unit to the left i.e. x becomes $x - 1$. Therefore the new equation will become:

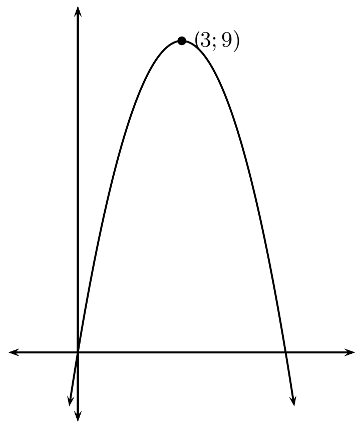
$$\begin{aligned} y &= (x-1)^2 - 2(x-1) - 3 \\ &= x^2 - 2x + 1 - 2x + 2 - 3 \\ &= x^2 - 4x \end{aligned}$$

If the given parabola is shifted 3 units down i.e. y becomes $y + 3$. The new equation will be: (Notice the x-axis then moves 3 units upwards)

$$\begin{aligned} y + 3 &= x^2 - 2x - 3 \\ y &= x^2 - 2x - 6 \end{aligned}$$

Chapter 11 End of Chapter Exercises

1. Show that if $a < 0$, then the range of $f(x) = a(x+p)^2+q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.
2. If $(2;7)$ is the turning point of $f(x) = -2x^2 - 4ax + k$, find the values of the constants a and k.
3. The graph in the figure is represented by the equation $f(x) = ax^2 + bx$. The coordinates of the turning point are $(3;9)$. Show that $a = -1$ and $b = 6$.



4. Given: $y = x^2 - 2x + 3$. Give the equation of the new graph originating if:
   (a) The graph of f is moved three units to the left.
   (b) The x-axis is moved down three units.
5. A parabola with turning point $(-1;-4)$ is shifted vertically by 4 units upwards. What are the coordinates of the turning point of the shifted parabola?

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(1.) 011j (2.) 011k (3.) 011m (4.) 011n (5.) 011p

Hyperbolic Functions and Graphs 12

12.1 Introduction

EMBBD

In the previous chapter, we discussed the graphs of general quadratic functions. In this chapter we will learn more about sketching and interpreting the graphs of general hyperbolic functions.

See introductory video: VMfmc at www.everythingmaths.co.za

12.2 Functions of the Form $y = \frac{a}{x+p} + q$

EMBBE

This form of the hyperbolic function is slightly more complex than the form studied in Grade 10.

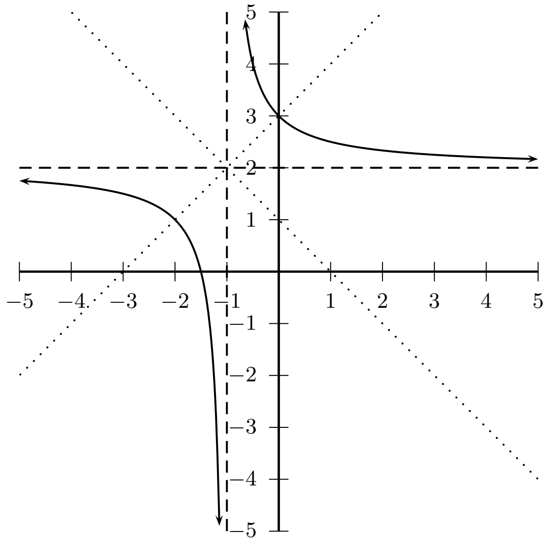


Figure 12.1: General shape and position of the graph of a function of the form $f(x) = \frac{a}{x+p} + q$. The asymptotes are shown as dashed lines.

Activity:	Functions of the Form $y = \frac{a}{x+p} + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = \frac{-2}{x+1} + 1$
   (b) $b(x) = \frac{-1}{x+1} + 1$
   (c) $c(x) = \frac{0}{x+1} + 1$
   (d) $d(x) = \frac{1}{x+1} + 1$
   (e) $e(x) = \frac{2}{x+1} + 1$

   Use your results to deduce the effect of a.
2. On the same set of axes, plot the following graphs:
   (a) $f(x) = \frac{1}{x-2} + 1$
   (b) $g(x) = \frac{1}{x-1} + 1$
   (c) $h(x) = \frac{1}{x+0} + 1$
   (d) $j(x) = \frac{1}{x+1} + 1$
   (e) $k(x) = \frac{1}{x+2} + 1$

   Use your results to deduce the effect of p.
3. Following the general method of the above activities, choose your own values of a and p to plot five different graphs of $y = \frac{a}{x+p} + q$ to deduce the effect of q.

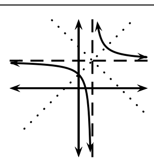
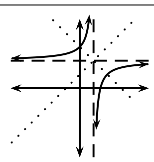
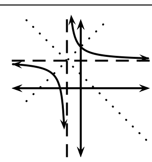
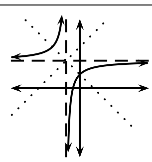
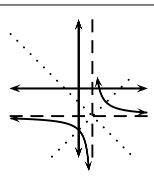
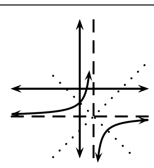
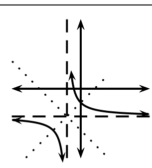
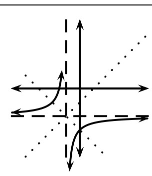
You should have found that the sign of a affects whether the graph is located in the first and third quadrants, or the second and fourth quadrants of Cartesian plane.

You should have also found that the value of p affects whether the x-intercept is negative ($p > 0$) or positive ($p < 0$).

You should have also found that the value of q affects whether the graph lies above the x-axis ($q > 0$) or below the x-axis ($q < 0$).

These different properties are summarised in Table 12.1. The axes of symmetry for each graph is shown as a dashed line.

Table 12.1: Table summarising general shapes and positions of functions of the form $y = \frac{a}{x+p} + q$. The axes of symmetry are shown as dashed lines.

	$p < 0$		$p > 0$	
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q > 0$				
$q < 0$				

Domain and Range

EMBBF

For $y = \frac{a}{x+p} + q$, the function is undefined for $x = -p$. The domain is therefore $\{x : x \in \mathbb{R}, x \neq -p\}$.

We see that $y = \frac{a}{x+p} + q$ can be re-written as:

$$\begin{aligned} y &= \frac{a}{x+p} + q \\ y - q &= \frac{a}{x+p} \\ \text{If } x \neq -p \text{ then: } (y-q)(x+p) &= a \\ x + p &= \frac{a}{y-q} \end{aligned}$$

This shows that the function is undefined at $y = q$. Therefore the range of $f(x) = \frac{a}{x+p} + q$ is $\{f(x) : f(x) \in R, f(x) \neq q\}$.

For example, the domain of $g(x) = \frac{2}{x+1} + 2$ is $\{x : x \in \mathbb{R}, x \neq -1\}$ because $g(x)$ is undefined at $x = -1$.

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ (y-2) &= \frac{2}{x+1} \\ (y-2)(x+1) &= 2 \\ (x+1) &= \frac{2}{y-2} \end{aligned}$$

We see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty; 2) \cup (2; \infty)\}$.

Exercise 12 - 1

1. Determine the range of $y = \frac{1}{x} + 1$.
2. Given: $f(x) = \frac{8}{x-8} + 4$. Write down the domain of f.
3. Determine the domain of $y = -\frac{8}{x+1} + 3$

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(1.) 01za (2.) 01zb (3.) 01zc

Intercepts

EMBBG

For functions of the form, $y = \frac{a}{x+p} + q$, the intercepts with the x and y axis are calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$y = \frac{a}{x+p} + q \quad (12.1)$$
$$y_{int} = \frac{a}{0+p} + q \quad (12.2)$$
$$= \frac{a}{p} + q \quad (12.3)$$

For example, the y-intercept of $g(x) = \frac{2}{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ y_{int} &= \frac{2}{0+1} + 2 \\ &= \frac{2}{1} + 2 \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

The x-intercepts are calculated by setting $y = 0$ as follows:

$$y = \frac{a}{x+p} + q \quad (12.4)$$
$$0 = \frac{a}{x_{int}+p} + q \quad (12.5)$$
$$\frac{a}{x_{int}+p} = -q \quad (12.6)$$
$$a = -q(x_{int}+p) \quad (12.7)$$
$$x_{int}+p = \frac{a}{-q} \quad (12.8)$$
$$x_{int} = \frac{a}{-q} - p \quad (12.9)$$

For example, the x-intercept of $g(x) = \frac{2}{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ 0 &= \frac{2}{x_{int}+1} + 2 \\ -2 &= \frac{2}{x_{int}+1} \\ -2(x_{int}+1) &= 2 \\ x_{int}+1 &= \frac{2}{-2} \\ x_{int} &= -1 - 1 \\ x_{int} &= -2 \end{aligned}$$

Exercise 12 - 2

1. Given: $h(x) = \frac{1}{x+4} - 2$. Determine the coordinates of the intercepts of h with the x- and y-axes.
2. Determine the x-intercept of the graph of $y = \frac{5}{x} + 2$. Give the reason why there is no y-intercept for this function.

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(1.) 011q (2.) 011r

Asymptotes

EMBBH

There are two asymptotes for functions of the form $y = \frac{a}{x+p} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = -p$ and for $y = q$. Therefore the asymptotes are $x = -p$ and $y = q$.

For example, the domain of $g(x) = \frac{2}{x+1} + 2$ is $\{x : x \in \mathbb{R}; x \neq -1\}$ because $g(x)$ is undefined at $x = -1$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty; 2) \cup (2; \infty)\}$.

From this we deduce that the asymptotes are at $x = -1$ and $y = 2$.

Exercise 12 - 3

1. Given: $h(x) = \frac{1}{x+4} - 2$.Determine the equations of the asymptotes of h.
2. Write down the equation of the vertical asymptote of the graph $y = \frac{1}{x-1}$.

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(1.) 01zd (2.) 01ze

Sketching Graphs of the Form $f(x) = \frac{a}{x+p} + q$

EMBBI

In order to sketch graphs of functions of the form, $f(x) = \frac{a}{x+p} + q$, we need to calculate four characteristics:

1. domain and range
2. asymptotes
3. y-intercept
4. x-intercept

For example, sketch the graph of $g(x) = \frac{2}{x+1} + 2$. Mark the intercepts and asymptotes.

We have determined the domain to be $\{x : x \in \mathbb{R}, x \neq -1\}$ and the range to be $\{g(x) : g(x) \in (-\infty; 2) \cup (2; \infty)\}$. Therefore the asymptotes are at $x = -1$ and $y = 2$. The y-intercept $= 4$ and the x-intercept $= -2$.

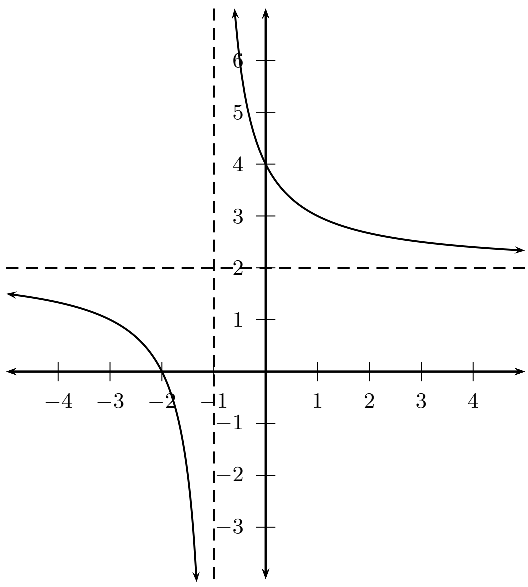

Figure 12.2: Graph of $g(x) = \frac{2}{x+1} + 2$.

Exercise 12 - 4

1. Draw the graph of $y = \frac{1}{x} + 2$. Indicate the horizontal asymptote.

2. Given: $h(x) = \frac{1}{x+4} - 2$. Sketch the graph of h showing clearly the asymptotes and ALL intercepts with the axes.

3. Draw the graph of $y = \frac{1}{x}$ and $y = -\frac{8}{x+1} + 3$ on the same system of axes.

4. Draw the graph of $y = \frac{5}{x-2,5} + 2$. Explain your method.

5. Draw the graph of the function defined by $y = \frac{8}{x-8} + 4$. Indicate the asymptotes and intercepts with the axes.

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(1.) 011s (2.) 011t (3.) 011u (4.) 011v (5.) 011w

Chapter 12 End of Chapter Exercises

1. Plot the graph of the hyperbola defined by $y = \frac{2}{x}$ for $-4 \leq x \leq 4$. Suppose the hyperbola is shifted 3 units to the right and 1 unit down. What is the new equation then?
2. Based on the graph of $y = \frac{1}{x}$, determine the equation of the graph with asymptotes $y = 2$ and $x = 1$ and passing through the point $(2; 3)$.

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(1.) 011x (2.) 011y

Exponential Functions and Graphs 13

13.1 Introduction

EMBBJ

Building on the previous two chapters, we will discuss the sketching and interpretation of the graphs of general exponential functions in this chapter.

See introductory video: VMfmg at www.everythingmaths.co.za

13.2 Functions of the Form $y = ab^{(x+p)} + q$ for $b > 0$

EMBBK

This form of the exponential function is slightly more complex than the form studied in Grade 10.

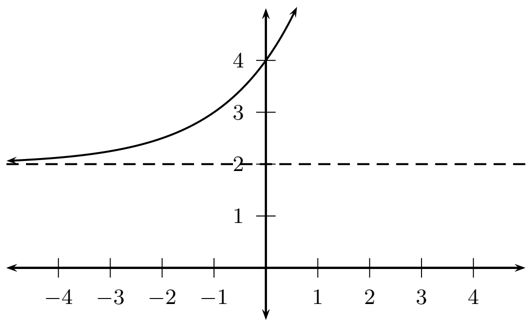

Figure 13.1: General shape and position of the graph of a function of the form $f(x) = ab^{(x+p)} + q$.

Activity:	*Functions of the Form $y = ab^{(x+p)} + q$*

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = -2^{(x+1)} + 1$
   (b) $b(x) = -1^{(x+1)} + 1$
   (c) $d(x) = 1^{(x+1)} + 1$
   (d) $e(x) = 2^{(x+1)} + 1$

   Use your results to deduce the effect of a.
2. On the same set of axes, plot the following graphs:
   (a) $f(x) = 2^{(x+1)} - 2$

(b) $g(x) = 2^{(x+1)} - 1$
(c) $h(x) = 2^{(x+1)} + 0$
(d) $j(x) = 2^{(x+1)} + 1$
(e) $k(x) = 2^{(x+1)} + 2$

Use your results to deduce the effect of q.

3. Following the general method of the above activities, choose your own values of a and q to plot five different graphs of $y = ab^{(x+p)} + q$ to deduce the effect of p.

You should have found that the value of a affects whether the graph is above the asymptote ($a > 0$) or below the asymptote ($a < 0$).

You should have also found that the value of p affects the position of the x-intercept.

You should have also found that the value of q affects the position of the y-intercept.

These different properties are summarised in Table 13.1. The axes of symmetry for each graph is shown as a dashed line.

Table 13.1: Table summarising general shapes and positions of functions of the form$y = ab^{(x+p)} + q$.

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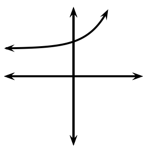
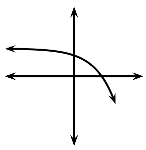
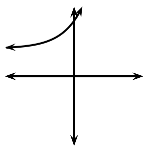
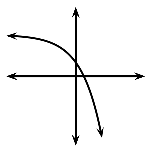
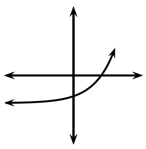
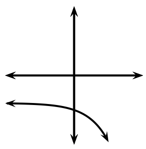
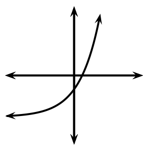
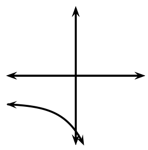
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q > 0$				
$q < 0$				

Domain and Range

EMBBL

For $y = ab^{(x+p)} + q$, the function is defined for all real values of x. Therefore, the domain is $\{x : x \in \mathbb{R}\}$.

The range of $y = ab^{(x+p)} + q$ is dependent on the sign of a.

If $a > 0$ then:

$$\begin{aligned} b^{(x+p)} &> 0 \\ a \,.\, b^{(x+p)} &> 0 \\ a \,.\, b^{(x+p)} + q &> q \\ f(x) &> q \end{aligned}$$

Therefore, if $a > 0$, then the range is $\{f(x) : f(x) \in [q;\infty)\}$.

If $a < 0$ then:

$$\begin{aligned} b^{(x+p)} &> 0 \\ a \,.\, b^{(x+p)} &< 0 \\ a \,.\, b^{(x+p)} + q &< q \\ f(x) &< q \end{aligned}$$

Therefore, if $a < 0$, then the range is $\{f(x) : f(x) \in (-\infty; q]\}$.

For example, the domain of $g(x) = 3 \,.\, 2^{x+1} + 2$ is $\{x : x \in \mathbb{R}\}$. For the range,

$$\begin{aligned} 2^{x+1} &> 0 \\ 3 \,.\, 2^{x+1} &> 0 \\ 3 \,.\, 2^{x+1} + 2 &> 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2; \infty)\}$.

Exercise 13 - 1

1. Give the domain of $y = 3^x$.
2. What is the domain and range of $f(x) = 2^x$?
3. Determine the domain and range of $y = (1{,}5)^{x+3}$.

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(1.) 011z (2.) 0120 (3.) 0121

Intercepts

EMBBM

For functions of the form, $y = ab^{(x+p)} + q$, the intercepts with the x- and y-axis are calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ab^{(x+p)} + q && (13.1) \\ y_{int} &= ab^{(0+p)} + q && (13.2) \\ &= ab^{p} + q && (13.3) \end{aligned}$$

For example, the y-intercept of $g(x) = 3 \,.\, 2^{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= 3 \,.\, 2^{x+1} + 2 \\ y_{int} &= 3 \,.\, 2^{0+1} + 2 \\ &= 3 \,.\, 2^{1} + 2 \\ &= 3 \,.\, 2 + 2 \\ &= 8 \end{aligned}$$

The x-intercepts are calculated by setting $y = 0$ as follows:

$$\begin{aligned} y &= ab^{(x+p)} + q && (13.4)\\ 0 &= ab^{(x_{int}+p)} + q && (13.5)\\ ab^{(x_{int}+p)} &= -q && (13.6)\\ b^{(x_{int}+p)} &= -\frac{q}{a} && (13.7) \end{aligned}$$

Which only has a real solution if either $a < 0$ or $q < 0$ and $a \neq 0$. Otherwise, the graph of the function of form $y = ab^{(x+p)} + q$ does not have any x-intercepts.

For example, the x-intercept of $g(x) = 3 \,.\, 2^{x+1} + 2$ is given by setting $y = 0$ to get:

$$\begin{aligned} y &= 3 \,.\, 2^{x+1} + 2\\ 0 &= 3 \,.\, 2^{x_{int}+1} + 2\\ -2 &= 3 \,.\, 2^{x_{int}+1}\\ 2^{x_{int}+1} &= \frac{-2}{3} \end{aligned}$$

which has no real solution. Therefore, the graph of $g(x) = 3 \,.\, 2^{x+1} + 2$ does not have a x-intercept.

Exercise 13 - 2

1. Give the y-intercept of the graph of $y = b^x + 2$.
2. Give the x- and y-intercepts of the graph of $y = \frac{1}{2}(1{,}5)^{x+3} - 0{,}75$.

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(1.) 0122 (2.) 0123

Asymptotes

EMBBN

The asymptote is the place at which the function is undefined. For functions of the form $y = ab^{(x+p)} + q$ this is along the line where $y = q$.

For example, the asymptote of $g(x) = 3 \,.\, 2^{x+1} + 2$ is $y = 2$.

Exercise 13 - 3

1. Give the equation of the asymptote of the graph of $y = 3^x - 2$.
2. What is the equation of the horizontal asymptote of the graph of $y = 3(0{,}8)^{x-1} - 3$?

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(1.) 0124 (2.) 0125

Sketching Graphs of the Form $f(x) = ab^{(x+p)} + q$

EMBBO

In order to sketch graphs of functions of the form, $f(x) = ab^{(x+p)} + q$, we need to determine four characteristics:

1. domain and range
2. y-intercept
3. x-intercept

For example, sketch the graph of $g(x) = 3 \,.\, 2^{x+1} + 2$. Mark the intercepts.

We have determined the domain to be $\{x : x \in \mathbb{R}\}$ and the range to be $\{g(x) : g(x) \in (2; \infty)\}$.

The y-intercept is $y_{int} = 8$ and there is no x-intercept.

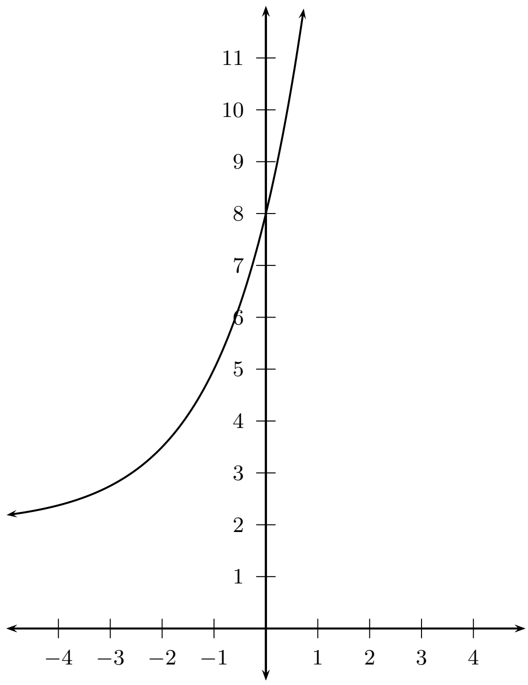

Figure 13.2: Graph of $g(x) = 3 \,.\, 2^{x+1} + 2$.

Exercise 13 - 4

1. Draw the graphs of the following on the same set of axes. Label the horizontal asymptotes and y-intercepts clearly.
   (a) $y = 2^x + 2$
   (b) $y = 2^{x+2}$

(c) $y = 2\,.\,2^x$

(d) $y = 2\,.\,2^{x+2} + 2$

2. Draw the graph of $f(x) = 3^x$.

3. Explain where a solution of $3^x = 5$ can be read off the graph.

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(1.) 01zf (2.) 01zg (3.) 01zh

Chapter 13 End of Chapter Exercises

1. The following table of values has columns giving the y-values for the graph $y = a^x$, $y = a^{x+1}$ and $y = a^x + 1$. Match a graph to a column.

x	A	B	C
-2	7,25	6,25	2,5
-1	3,5	2,5	1
0	2	1	0,4
1	1,4	0,4	0,16
2	1,16	0,16	0,064

2. The graph of $f(x) = 1 + a.2^x$ (a is a constant) passes through the origin.
   (a) Determine the value of a.
   (b) Determine the value of $f(-15)$ correct to five decimal places.
   (c) Determine the value of x, if $P(x; 0{,}5)$ lies on the graph of f.
   (d) If the graph of f is shifted 2 units to the right to give the function h, write down the equation of h.
3. The graph of $f(x) = a.b^x$ $(a \neq 0)$ has the point $P(2; 144)$ on f.
   (a) If $b = 0{,}75$, calculate the value of a.
   (b) Hence write down the equation of f.
   (c) Determine, correct to two decimal places, the value of $f(13)$.
   (d) Describe the transformation of the curve of f to h if $h(x) = f(-x)$.

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(1.) 0127 (2.) 0128 (3.) 0129

Gradient at a Point 14

14.1 Introduction EMBBP

In Grade 10, we investigated the idea of *average gradient* and saw that the gradients of most functions varied over different intervals. In Grade 11, we discuss the concept of average gradient further, and introduce the idea of the gradient of a curve at a point.

See introductory video: VMfns at www.everythingmaths.co.za

14.2 Average Gradient EMBBQ

We saw that the average gradient between two points on a curve is the gradient of the straight line passing through the two points.

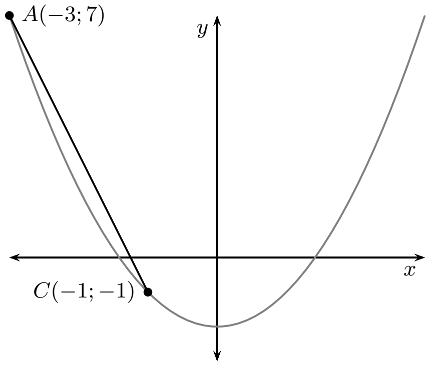


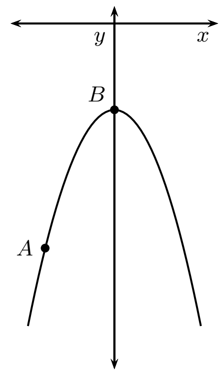
Figure 14.1: The average gradient between two points on a curve is the gradient of the straight line that passes through the points.

What happens to the gradient if we fix the position of one point and move the second point closer to the fixed point?

Activity: *Gradient at a Single Point on a Curve*

The curve shown below is defined by $y = -2x^2 - 5$. Point B is fixed at coordinates $(0;-5)$. The position of point A varies. Complete the table below by calculating the y-coordinates of point A for the given x-coordinates and then calculate the average gradient between points A and B.

x_A	y_A	average gradient
-2		
-1.5		
-1		
-0.5		
0		
0.5		
1		
1.5		
2		



What happens to the average gradient as A moves towards B? What happens to the average gradient as A moves away from B? What is the average gradient when A overlaps with B?

In Figure 14.2, the gradient of the straight line that passes through points A and C changes as A moves closer to C. At the point when A and C overlap, the straight line only passes through one point on the curve. Such a line is known as a tangent to the curve.

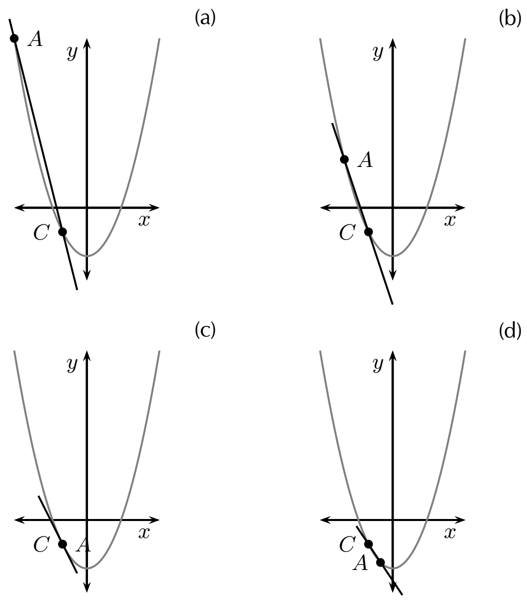


Figure 14.2: The gradient of the straight line between A and C changes as the point A moves along the curve towards C. There comes a point when A and C overlap (as shown in (c)). At this point the line is a tangent to the curve.

We therefore introduce the idea of a gradient at a single point on a curve. The gradient at a point on a curve is simply the gradient of the tangent to the curve at the given point.

Example 1: *Average Gradient*

QUESTION

Find the average gradient between two points $P(a; g(a))$ and $Q(a + h; g(a + h))$ on a curve $g(x) = x^2$. Then find the average gradient between $P(2; g(2))$ and $Q(4; g(4))$. Finally, explain what happens to the average gradient if P moves closer to Q.

SOLUTION

Step 1 : ***Label x points***

$$x_1 = a$$
$$x_2 = a + h$$

Step 2 : ***Determine y coordinates***

Using the function $g(x) = x^2$, we can determine:

$$y_1 = g(a) = a^2$$

$$\begin{aligned} y_2 &= g(a+h) \\ &= (a+h)^2 \\ &= a^2 + 2ah + h^2 \end{aligned}$$

Step 3 : ***Calculate average gradient***

$$\begin{aligned} \frac{y_2 - y_1}{x_2 - x_1} &= \frac{(a^2 + 2ah + h^2) - (a^2)}{(a+h) - (a)} \\ &= \frac{a^2 + 2ah + h^2 - a^2}{a + h - a} \\ &= \frac{2ah + h^2}{h} \\ &= \frac{h(2a + h)}{h} \\ &= 2a + h \end{aligned} \qquad (14.1)$$

The average gradient between $P(a; g(a))$ and $Q(a + h; g(a + h))$ on the curve $g(x) = x^2$ is $2a + h$.

Step 4 : ***Calculate the average gradient between $P(2; g(2))$ and $Q(4; g(4))$***

We can use the result in (14.1), but we have to determine what a and h are. We do this by looking at the definitions of P and Q. The x-coordinate of P is a and the x-coordinate of Q is $a + h$ therefore if we assume that $a = 2$ and $a + h = 4$, then $h = 2$.

Then the average gradient is:

$$2a + h = 2(2) + (2) = 6$$

Step 5 : ***When P moves closer to Q***

When point P moves closer to point Q, h gets smaller. This means that the average gradient also gets smaller. When the point Q overlaps with the point P $h = 0$ and the average gradient is given by $2a$.

We now see that we can write the equation to calculate average gradient in a slightly different manner. If we have a curve defined by $f(x)$ then for two points P and Q with $P(a; f(a))$ and $Q(a+h; f(a+h))$, then the average gradient between P and Q on $f(x)$ is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{(a+h) - (a)}$$
$$= \frac{f(a+h) - f(a)}{h}$$

This result is important for calculating the gradient at a point on a curve and will be explored in greater detail in Grade 12.

Chapter 14 End of Chapter Exercises

1. (a) Determine the average gradient of the curve $f(x) = x(x + 3)$ between $x = 5$ and $x = 3$.
   (b) Hence, state what you can deduce about the function f between $x = 5$ and $x = 3$.
2. A(1;3) is a point on $f(x) = 3x^2$.
   (a) Determine the gradient of the curve at point A.
   (b) Hence, determine the equation of the tangent line at A.
3. Given: $f(x) = 2x^2$.
   (a) Determine the average gradient of the curve between $x = -2$ and $x = 1$.
   (b) Determine the gradient of the curve of f where $x = 2$.

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(1.) 012a (2.) 012b (3.) 012c

Linear Programming 15

15.1 Introduction EMBBR

In everyday life people are interested in knowing the most efficient way of carrying out a task or achieving a goal. For example, a farmer might want to know how many crops to plant during a season in order to maximise yield (produce) or a stock broker might want to know how much to invest in stocks in order to maximise profit. These are examples of **optimisation** problems, where by optimising we mean finding the maxima or minima of a function.

See introductory video: VMfnt at www.everythingmaths.co.za

15.2 Terminology EMBBS

There are some basic terms which you need to become familiar with for the linear programming chapters.

Decision Variables EMBBT

The aim of an optimisation problem is to find the values of the decision variables. These values are unknown at the beginning of the problem. Decision variables usually represent things that can be changed, for example the rate at which water is consumed or the number of birds living in a certain park.

Objective Function EMBBU

The objective function is a mathematical combination of the decision variables and represents the function that we want to optimise (i.e. maximise or minimise). We will only be looking at objective functions which are functions of two variables. For example, in the case of the farmer, the objective function is the yield and it is dependent on the amount of crops planted. If the farmer has two crops then the objective function $f(x,y)$ is the yield, where x represents the amount of the first crop planted and y represents the amount of the second crop planted. For the stock broker, assuming that there are two stocks to invest in, the objective function $f(x,y)$ is the amount of profit earned by investing x rand in the first stock and y rand in the second.

Constraints

EMBBV

Constraints, or **restrictions**, are often placed on the variables being optimised. For the example of the farmer, he cannot plant a negative number of crops, therefore the constraints would be:

$$x \geq 0$$
$$y \geq 0.$$

Other constraints might be that the farmer cannot plant more of the second crop than the first crop and that no more than 20 units of the first crop can be planted. These constraints can be written as:

$$x \geq y$$
$$x \leq 20$$

Constraints that have the form

$$ax + by \leq c$$

or

$$ax + by = c$$

are called **linear** constraints. Examples of linear constraints are:

$$x + y \leq 0$$
$$-2x = 7$$
$$y \leq \sqrt{2}$$

Feasible Region and Points

EMBBW

Tip

The constraints are used to create bounds of the solution.

Tip

Points that satisfy the constraints are called feasible solutions.

Constraints mean that we cannot just take any x and y when looking for the x and y that optimise our objective function. If we think of the variables x and y as a point (x,y) in the xy-plane then we call the set of all points in the xy-plane that satisfy our constraints the **feasible region**. Any point in the feasible region is called a **feasible point**.

For example, the constraints

$$x \geq 0$$
$$y \geq 0.$$

mean that only values of x and y that are positive are allowed. Similarly, the constraint

$$x \geq y$$

means that only values of x that are greater than or equal to the y values are allowed.

$$x \leq 20$$

means that only x values which are less than or equal to 20 are allowed.

The Solution

EMBBX

Once we have determined the feasible region the **solution** of our problem will be the feasible point where the objective function is a maximum / minimum. Sometimes there will be more than one feasible point where the objective function is a maximum/minimum — in this case we have more than one solution.

15.3 Example of a Problem

EMBBY

A simple problem that can be solved with linear programming involves Mrs Nkosi and her farm.

> Mrs Nkosi grows mielies and potatoes on a farm of $100\ \text{m}^2$. She has accepted orders that will need her to grow at least $40\ \text{m}^2$ of mielies and at least $30\ \text{m}^2$ of potatoes. Market research shows that the demand this year will be at least twice as much for mielies as for potatoes and so she wants to use at least twice as much area for mielies as for potatoes. She expects to make a profit of R650 per m^2 for her mielies and R$1\ 500$ per m^2 on her potatoes. How should she divide her land so that she can earn the most profit?

Let q represent the area of mielies grown and let p be the area of potatoes grown.

We shall see below how we can solve this problem.

15.4 Method of Linear Programming

EMBBZ

Method: Linear Programming

EMBCA

1. Identify the decision variables in the problem.
2. Write constraint equations
3. Write objective function as an equation
4. Solve the problem

15.5 Skills You Will Need

EMBCB

Writing Constraint Equations

EMBCC

You will need to be comfortable with converting a word description to a mathematical description for linear programming. Some of the words that are used is summarised in Table 15.1.

Table 15.1: Phrases and mathematical equivalents.

Words	Mathematical description
x equals a	$x = a$
x is greater than a	$x > a$
x is greater than or equal to a	$x \geq a$
x is less than a	$x < a$
x is less than or equal to a	$x \leq a$
x must be at least a	$x \geq a$
x must be at most a	$x \leq a$

Example 1: *Writing constraints as equations*

QUESTION

Mrs Nkosi grows mielies and potatoes on a farm of $100\ \text{m}^2$. She has accepted orders that will need her to grow at least $40\ \text{m}^2$ of mielies and at least $30\ \text{m}^2$ of potatoes. Market research shows that the demand this year will be at least twice as much for mielies as for potatoes and so she wants to use at least twice as much area for mielies as for potatoes.

SOLUTION

Step 1 : ***Identify the decision variables***

There are two decision variables: the area used to plant mielies (q) and the area used to plant potatoes (p).

Step 2 : ***Identify the phrases that constrain the decision variables***

- grow at least $40\ \text{m}^2$ of mielies
- grow at least $30\ \text{m}^2$ of potatoes
- area of farm is $100\ \text{m}^2$
- demand is at least twice as much for mielies as for potatoes

Step 3 : ***For each phrase, write a constraint***

- $q \geq 40$
- $p \geq 30$
- $q + p \leq 100$
- $q \geq 2p$

Exercise 15 - 1

Write the following constraints as equations:

1. Michael is registering for courses at university. Michael needs to register for at least 4 courses.
2. Joyce is also registering for courses at university. She cannot register for more than 7 courses.
3. In a geography test, Simon is allowed to choose 4 questions from each section.
4. A baker can bake at most 50 chocolate cakes in one day.
5. Megan and Katja can carry at most 400 koeksisters.

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(1.) 012d (2.) 012e (3.) 012f (4.) 012g (5.) 012h

Writing the Objective Function

EMBCD

If the objective function is not given to you as an equation, you will need to be able to convert a word description to an equation to get the objective function.

You will need to look for words like:

- most profit
- least cost
- largest area

Example 2: *Writing the objective function*

QUESTION

The cost of hiring a small trailer is R500 per day and the cost of hiring a big trailer is R800 per day. Write down the objective function that can be used to find the cheapest cost for hiring trailers for one day.

SOLUTION

Step 1 : ***Identify the decision variables***
There are two decision variables:the number of small trailers (m) and the number of big trailers (n).

Step 2 : ***Write the purpose of the objective function***
The purpose of the objective function is to minimise cost.

Step 3 : ***Write the objective function***
The cost of hiring m small trailers for one day is:

$$500 \times m$$

The cost of hiring n big trailers for one day is:

$$800 \times n$$

Therefore the objective function, which is the total cost of hiring m small trailers and n big trailers for one day is:

$$(500 \times m) + (800 \times n)$$

Example 3: *Writing the objective function*

QUESTION

Mrs Nkosi expects to make a profit of R650 per m^2 for her mielies and R1 500 per m^2 on her potatoes. How should she divide her land so that she can earn the most profit?

SOLUTION

Step 1 : ***Identify the decision variables***
There are two decision variables: the area used to plant mielies (q) and the area used to plant potatoes (p).

Step 2 : ***Write the purpose of the objective function***
The purpose of the objective function is to maximise profit.

Step 3 : ***Write the objective function***
The profit of planting $q \text{ m}^2$ of mielies is:

$$650 \times q$$

The profit of planting $p \text{ m}^2$ of potatoes is:

$$1\ 500 \times p$$

Therefore the objective function, which is the total profit of planting mielies and potatoes is:

$$(650 \times q) + (1\ 500 \times p)$$

Exercise 15 - 2

1. The *EduFurn* furniture factory manufactures school chairs and school desks. They make a profit of R50 on each chair sold and of R60 on each desk sold. Write an equation that will show how much profit they will make by selling the chairs and desks.
2. A manufacturer makes small screen GPS units and wide screen GPS units. If the profit on a small screen GPS unit is R500 and the profit on a wide screen GPS unit is R250, write an equation that will show the possible maximum profit.

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(1.) 012i (2.) 012j

Solving the Problem *EMBCE*

The numerical method involves using the points along the boundary of the feasible region, and determining which point optimises the objective function.

Activity: *Numerical Method*

Use the objective function

$$(650 \times q) + (1\ 500 \times p)$$

to calculate Mrs Nkosi's profit for the following feasible solutions:

q	p	Profit
60	30	
65	30	
70	30	
$66\frac{2}{3}$	$33\frac{1}{3}$	

The question is *how do you find the feasible region?* We will use the graphical method of solving a system of linear equations to determine the feasible region. We draw all constraints as graphs and mark the area that satisfies all constraints. This is shown in Figure 15.1 for Mrs Nkosi's farm.

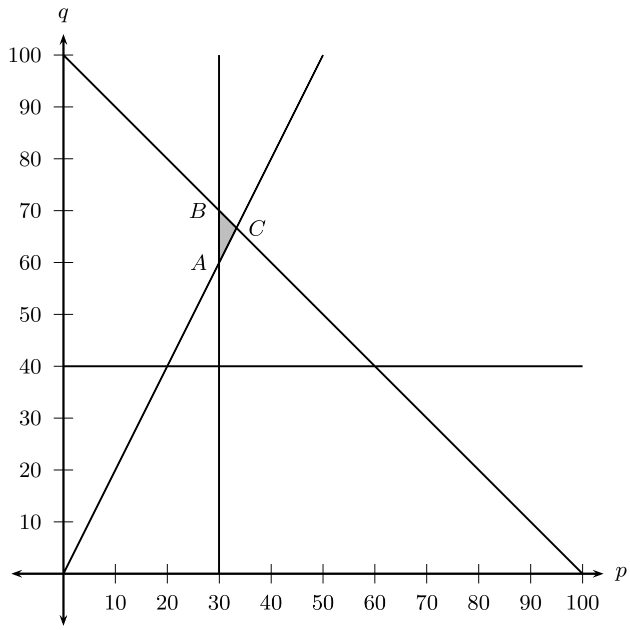


Figure 15.1: Graph of the feasible region

Vertices (singular: vertex) are the points on the graph where two or more of the constraints overlap or cross. If the linear objective function has a minimum or maximum value, it will occur at one or more of the vertices of the feasible region.

Now we can use the methods we learnt previously to find the points at the vertices of the feasible region. In Figure 15.1, vertex A is at the intersection of $p = 30$ and $q = 2p$. Therefore, the coordinates of A are $(30; 60)$. Similarly vertex B is at the intersection of $p = 30$ and $q = 100 - p$. Therefore the coordinates of B are $(30; 70)$. Vertex C is at the intersection of $q = 100 - p$ and $q = 2p$, which gives $(33\frac{1}{3}; 66\frac{2}{3})$ for the coordinates of C.

If we now substitute these points into the objective function, we get the following:

q	p	Profit
60	30	81 000
70	30	87 000
$66\frac{2}{3}$	$33\frac{1}{3}$	89 997

Therefore Mrs Nkosi makes the most profit if she plants $66\frac{2}{3}\ \text{m}^2$ of mielies and $33\frac{1}{3}\ \text{m}^2$ of potatoes. Her profit is R89 997.

Example 4: *Prizes!*

QUESTION

As part of their opening specials, a furniture store has promised to give away at least 40 *prizes with a total value of at least R*$2\ 000$*. The prizes are kettles and toasters.*

1. *If the company decides that there will be at least* 10 *of each prize, write down two more inequalities from these constraints.*
2. *If the cost of manufacturing a kettle is R*60 *and a toaster is R*50*, write down an objective function* C *which can be used to determine the cost to the company of both kettles and toasters.*
3. *Sketch the graph of the feasibility region that can be used to determine all the possible combinations of kettles and toasters that honour the promises of the company.*
4. *How many of each prize will represent the cheapest option for the company?*
5. *How much will this combination of kettles and toasters cost?*

SOLUTION

Step 1 : ***Identify the decision variables***
Let the number of kettles be x and the number of toasters be y and write down two constraints apart from $x \geq 0$ and $y \geq 0$ that must be adhered to.

Step 2 : ***Write constraint equations***
Since there will be at least 10 of each prize we can write:

$$x \geq 10$$

and

$$y \geq 10$$

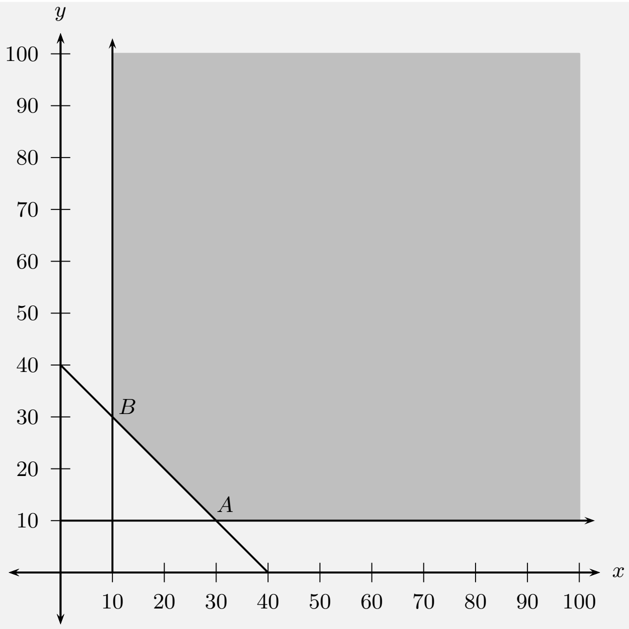
Also the store has promised to give away at least 40 prizes in total. Therefore:

$$x + y \geq 40$$

Step 3 : ***Write the objective function***
The cost of manufacturing a kettle is R60 and a toaster is R50. Therefore the cost the total cost C is:

$$C = 60x + 50y$$

Step 4 : ***Sketch the graph of the feasible region***



Step 5 : ***Determine vertices of feasible region***
From the graph, the coordinates of vertex A are $(30;10)$ and the coordinates of vertex B are $(10;30)$.

Step 6 : ***Calculate cost at each vertex***
At vertex A, the cost is:

$$\begin{aligned} C &= 60x + 50y \\ &= 60(30) + 50(10) \\ &= 1\ 800 + 500 \\ &= 2\ 300 \end{aligned}$$

At vertex B, the cost is:

$$\begin{aligned} C &= 60x + 50y \\ &= 60(10) + 50(30) \\ &= 600 + 1\ 500 \\ &= 2\ 100 \end{aligned}$$

Step 7 : ***Write the final answer***
The cheapest combination of prizes is 10 kettles and 30 toasters, costing the company R2 100.

Chapter 15 End of Chapter Exercises

1. You are given a test consisting of two sections. The first section is on algebra and the second section is on geometry. You are not allowed to answer more than 10 questions from any section, but you have to answer at least 4 algebra questions. The time allowed is not more than 30 minutes. An algebra problem will take 2 minutes and a geometry problem will take 3 minutes to solve.

   If you answer x algebra questions and y geometry questions,

   (a) Formulate the constraints which satisfy the above constraints.

   (b) The algebra questions carry 5 marks each and the geometry questions carry 10 marks each. If T is the total marks, write down an expression for T.

2. A local clinic wants to produce a guide to healthy living. The clinic intends to produce the guide in two formats: a short video and a printed book. The clinic needs to decide how many of each format to produce for sale. Estimates show that no more than $10\ 000$ copies of both items together will be sold. At least $4\ 000$ copies of the video and at least $2\ 000$ copies of the book could be sold, although sales of the book are not expected to exceed $4\ 000$ copies. Let x be the number of videos sold, and y the number of printed books sold.

   (a) Write down the constraint inequalities that can be deduced from the given information.

   (b) Represent these inequalities graphically and indicate the feasible region clearly.

   (c) The clinic is seeking to maximise the income, I, earned from the sales of the two products. Each video will sell for R50 and each book for R30. Write down the objective function for the income.

   (d) What maximum income will be generated by the two guides?

3. A patient in a hospital needs at least 18 grams of protein, $0{,}006$ grams of vitamin C and $0{,}005$ grams of iron per meal, which consists of two types of food, A and B. Type A contains 9 grams of protein, $0{,}002$ grams of vitamin C and no iron per serving. Type B contains 3 grams of protein, $0{,}002$ grams of vitamin C and $0{,}005$ grams of iron per serving. The energy value of A is 800 kilojoules and of B 400 kilojoules per serving. A patient is not allowed to have more than 4 servings of A and 5 servings of B. There are x servings of A and y servings of B on the patient's plate.

   (a) Write down in terms of x and y

      i. The mathematical constraints which must be satisfied.

      ii. The kilojoule intake per meal.

   (b) Represent the constraints graphically on graph paper. Use the scale 1 unit $=$ 20mm on both axes. Shade the feasible region.

   (c) Deduce from the graphs, the values of x and y which will give the minimum kilojoule intake per meal for the patient.

4. A certain motorcycle manufacturer produces two basic models, the *Super X* and the *Super Y*. These motorcycles are sold to dealers at a profit of R20 000 per *Super X* and R10 000 per *Super Y*. A *Super X* requires 150 hours for assembly, 50 hours for painting and finishing and 10 hours for checking and testing. The *Super Y* requires 60 hours for assembly, 40 hours for painting and finishing and 20 hours for checking and testing. The total number of hours available per month is: $30\ 000$ in the assembly department, $13\ 000$ in the painting and finishing department and $5\ 000$ in the checking and testing department.

   The above information can be summarised by the following table:

Department	Hours for *Super X*	Hours for *Super Y*	Maximum hours available per month
Assembly	150	60	30 000
Painting and Finishing	50	40	13 000
Checking and Testing	10	20	5 000

Let x be the number of *Super X* and y be the number of *Super Y* models manufactured per month.

(a) Write down the set of constraint inequalities.

(b) Use the graph paper provided to represent the constraint inequalities.

(c) Shade the feasible region on the graph paper.

(d) Write down the profit generated in terms of x and y.

(e) How many motorcycles of each model must be produced in order to maximise the monthly profit?

(f) What is the maximum monthly profit?

5. A group of students plan to sell x hamburgers and y chicken burgers at a rugby match. They have meat for at most 300 hamburgers and at most 400 chicken burgers. Each burger of both types is sold in a packet. There are 500 packets available. The demand is likely to be such that the number of chicken burgers sold is at least half the number of hamburgers sold.

(a) Write the constraint inequalities.

(b) Two constraint inequalities are shown on the graph paper provided. Represent the remaining constraint inequalities on the graph paper.

(c) Shade the feasible region on the graph paper.

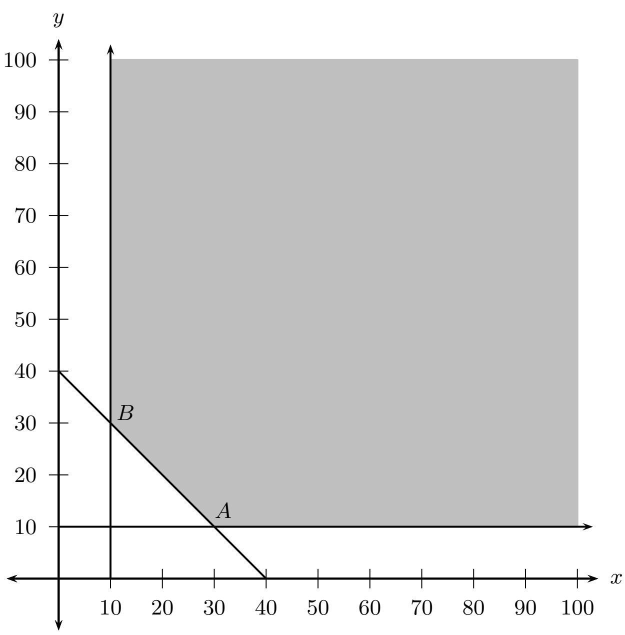
(d) A profit of R3 is made on each hamburger sold and R2 on each chicken burger sold. Write the equation which represents the total profit P in terms of x and y.

(e) The objective is to maximise profit. How many, of each type of burger, should be sold to maximise profit?

6. *Fashion-cards* is a small company that makes two types of cards, type X and type Y. With the available labour and material, the company can make not more than 150 cards of type X and not more than 120 cards of type Y per week. Altogether they cannot make more than 200 cards per week.

There is an order for at least 40 type X cards and 10 type Y cards per week. *Fashion-cards* makes a profit of R5 for each type X card sold and R10 for each type Y card.

Let the number of type X cards be x and the nu



mber of type Y cards be y, manufactured per week.

(a) One of the constraint inequalities which represents the restrictions above is $x \leq 150$. Write the other constraint inequalities.

(b) Represent the constraints graphically and shade the feasible region.

(c) Write the equation that represents the profit P (the objective function), in terms of x and y.

(d) Calculate the maximum weekly profit.

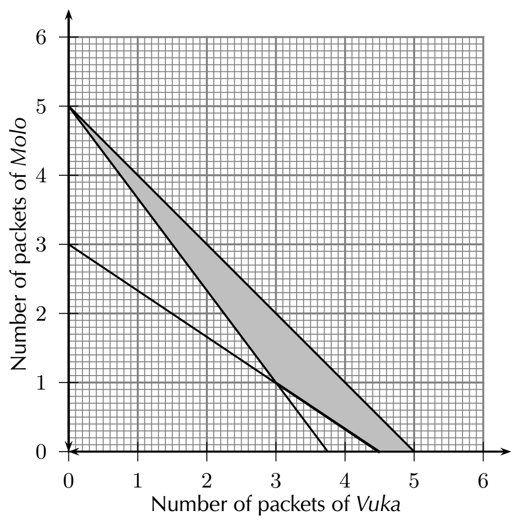
7. To meet the requirements of a specialised diet a meal is prepared by mixing two types of cereal, *Vuka* and *Molo*. The mixture must contain x packets of *Vuka* cereal and y packets of *Molo* cereal. The meal requires at least 15 g of protein and at least 72 g of carbohydrates. Each packet of *Vuka* cereal contains 4 g of protein and 16 g of carbohydrates. Each packet of *Molo* cereal contains 3 g of protein and 24 g of carbohydrates. There are at most 5 packets of cereal available. The feasible region is shaded on the attached graph paper.

   (a) Write down the constraint inequalities.

   (b) If *Vuka* cereal costs R6 per packet and *Molo* cereal also costs R6 per packet, use the graph to determine how many packets of each cereal must be used for the mixture to satisfy the above constraints in each of the following cases:

      i. The total cost is a minimum.

      ii. The total cost is a maximum (give all possibilities).



8. A bicycle manufacturer makes two different models of bicycles, namely mountain bikes and speed bikes. The bicycle manufacturer works under the following constraints:
No more than 5 mountain bicycles can be assembled daily.
No more than 3 speed bicycles can be assembled daily.
It takes one man to assemble a mountain bicycle, two men to assemble a speed bicycle and there are 8 men working at the bicycle manufacturer.
Let x represent the number of mountain bicycles and let y represent the number of speed bicycles.

   (a) Determine algebraically the constraints that apply to this problem.

   (b) Represent the constraints graphically on the graph paper.

   (c) By means of shading, clearly indicate the feasible region on the graph.

   (d) The profit on a mountain bicycle is R200 and the profit on a speed bicycle is R600. Write down an expression to represent the profit on the bicycles.

   (e) Determine the number of each model bicycle that would maximise the profit to the manufacturer.

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(1.) 012k (2.) 012m (3.) 012n (4.) 012p (5.) 012q (6.) 012r
(7.) 012s (8.) 012t

Geometry 16

16.1 Introduction EMBCF

Geometry is a good subject for learning not just about the mathematics of two and three-dimensional shapes, but also about how we construct mathematical arguments. In this chapter you will learn how to prove geometric theorems and discover some of the properties of shapes through small logical steps.

See introductory video: VMfqd at www.everythingmaths.co.za

16.2 Right Pyramids, Right Cones and Spheres EMBCG

A pyramid is a geometric solid that has a polygon base and the base is joined to a point, called the apex. Two examples of pyramids are shown in the left-most and centre figures in Figure 16.1. The right-most figure has an apex which is joined to a circular base and this type of geometric solid is called a cone. Cones are similar to pyramids except that their bases are circles instead of polygons.

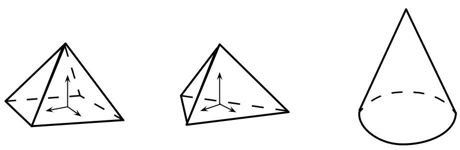


Figure 16.1: Examples of a square pyramid, a triangular pyramid and a cone.

Surface Area of a Pyramid

The surface area of a pyramid is calculated by adding the area of each face together.

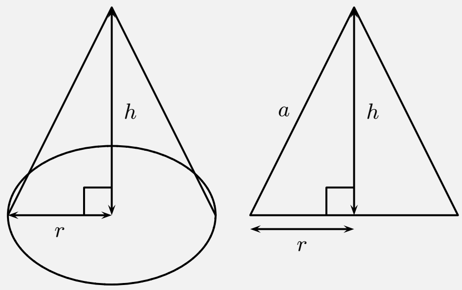
Example 1: *Surface Area*

QUESTION

If a cone has a height of h and a base of radius r, show that the surface area is $\pi r^2 + \pi r\sqrt{r^2 + h^2}$.

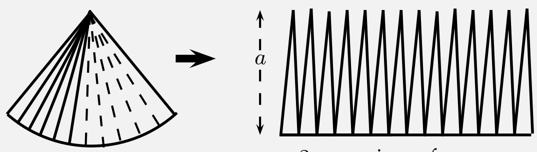
SOLUTION

Step 1 : ***Draw a picture***



Step 2 : ***Identify the faces that make up the cone***

The cone has two faces: the base and the walls. The base is a circle of radius r and the walls can be opened out to a sector of a circle.



$2\pi r$ = circumference

This curved surface can be cut into many thin triangles with height close to a (a is called the *slant height*). The area of these triangles will add up to $\frac{1}{2}\times$base$\times$height(of a small triangle) which is $\frac{1}{2} \times 2\pi r \times a = \pi ra$

Step 3 : ***Calculate*** a

a can be calculated by using the Theorem of Pythagoras. Therefore:

$$a = \sqrt{r^2 + h^2}$$

Step 4 : ***Calculate the area of the circular base***

$$A_b = \pi r^2$$

Step 5 : ***Calculate the area of the curved walls***

$$\begin{aligned} A_w &= \pi ra \\ &= \pi r\sqrt{r^2 + h^2} \end{aligned}$$

Step 6 : ***Calculate surface area*** A

$$\begin{aligned} A &= A_b + A_w \\ &= \pi r^2 + \pi r\sqrt{r^2 + h^2} \end{aligned}$$

Volume of a Pyramid: The volume of a pyramid is found by:

$$V = \frac{1}{3}A\,.\,h$$

where A is the area of the base and h is the perpendicular height.

A cone is like a pyramid, so the volume of a cone is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

A square pyramid has volume

$$V = \frac{1}{3}a^2 h$$

where a is the side length of the square base.

See video: VMfqj at www.everythingmaths.co.za

Example 2: *Volume of a Pyramid*

QUESTION

What is the volume of a square pyramid, 3 cm high with a side length of 2 cm?

SOLUTION

Step 1 : ***Determine the correct formula***

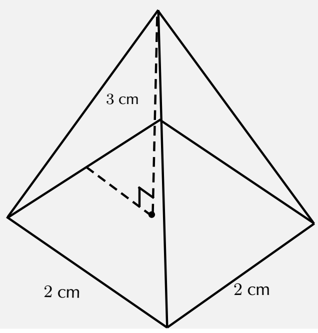
The volume of a pyramid is

$$V = \frac{1}{3}A\,.\,h,$$

where A is the area of the base and h is the height of the pyramid. For a square base this means

$$V = \frac{1}{3}a\,.\,a\,.\,h$$

where a is the length of the side of the square base.



Step 2 : ***Substitute the given values***

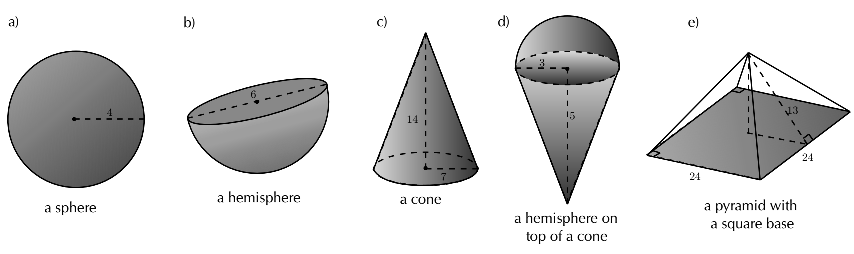
$$
\begin{aligned}
&= \frac{1}{3} \, . \, 2 \, . \, 2 \, . \, 3 \\
&= \frac{1}{3} \, . \, 12 \\
&= 4 \text{ cm}^3
\end{aligned}
$$

We accept the following formulae for volume and surface area of a sphere (ball).

$$
\begin{aligned}
\text{Surface area} &= 4\pi r^2 \\
\text{Volume} &= \frac{4}{3}\pi r^3
\end{aligned}
$$

Exercise 16 - 1

1. Calculate the volumes and surface areas of the following solids: (Hint for (e): find the perpendicular height using Pythagoras).



2. Water covers approximately 71% of the Earth's surface. Taking the radius of the Earth to be 6378 km, what is the total area of land (area not covered by water)?

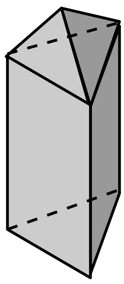
3.

A triangular pyramid is placed on top of a triangular prism. The prism has an equilateral triangle of side length 20 cm as a base, and has a height of 42 cm. The pyramid has a height of 12 cm.

(a) Find the total volume of the object.

(b) Find the area of each face of the pyramid.

(c) Find the total surface area of the object.



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(1.) 012u (2.) 012v (3.) 012w

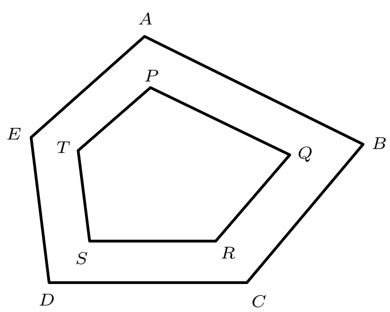
16.3 Similarity of Polygons

EMBCH

In order for two polygons to be similar the following must be true:

1. All corresponding angles must be congruent.

2. All corresponding sides must be in the same proportion to each other. Refer to the picture below: this means that the ratio of side AE on the large polygon to the side PT on the small polygon must be the same as the ratio of side AB to side PQ, BC/QR etc. for *all* the sides.



If

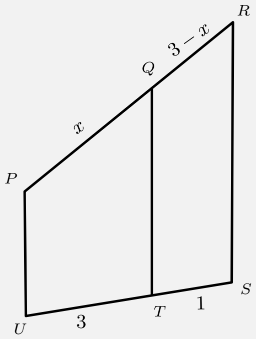
1. $\hat{A} = \hat{P}$; $\hat{B} = \hat{Q}$; $\hat{C} = \hat{R}$; $\hat{D} = \hat{S}$; $\hat{E} = \hat{T}$
   and
2. $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$

then the polygons $ABCDE$ and $PQRST$ are similar.

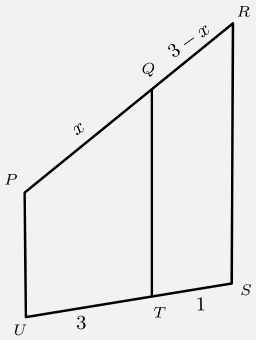
Example 3: *Similarity of Polygons*

QUESTION

Polygons $PQTU$ and $PRSU$ are similar. Find the value of x.



Polygons $PQTU$ and $PRSU$ are similar. Find the value of x.



SOLUTION

Step 1 : ***Identify corresponding sides***

Since the polygons are similar,

$$\begin{aligned} \frac{PQ}{PR} &= \frac{TU}{SU} \\ \therefore \frac{x}{x+(3-x)} &= \frac{3}{4} \\ \therefore \frac{x}{3} &= \frac{3}{4} \\ \therefore x &= \frac{9}{4} \end{aligned}$$

16.4 Triangle Geometry

EMBCI

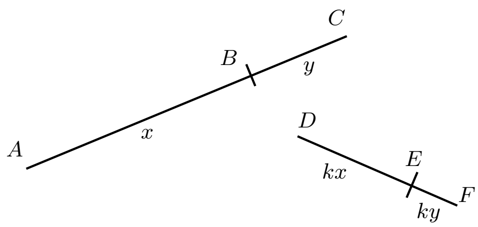
Proportion

EMBCJ

Two line segments are divided in the *same* proportion if the ratios between their parts are equal.

$$\frac{AB}{BC} = \frac{x}{y} = \frac{kx}{ky} = \frac{DE}{EF}$$

$\therefore$ the line segments are in the same proportion



If the line segments are proportional, the following also hold:

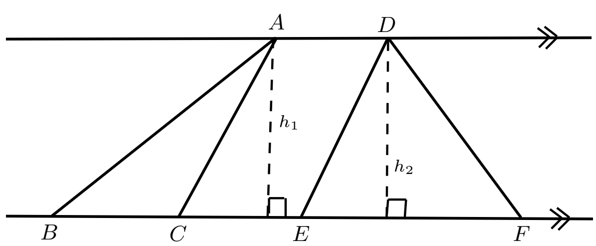
1. $\frac{CB}{AC} = \frac{FE}{DF}$
2. $AC\,.\,FE = CB\,.\,DF$
3. $\frac{AB}{BC} = \frac{DE}{FE}$ and $\frac{BC}{AB} = \frac{FE}{DE}$
4. $\frac{AB}{AC} = \frac{DE}{DF}$ and $\frac{AC}{AB} = \frac{DF}{DE}$

Proportionality of triangles

Triangles with equal heights have areas which are in the same proportion to each other as the bases of the triangles.

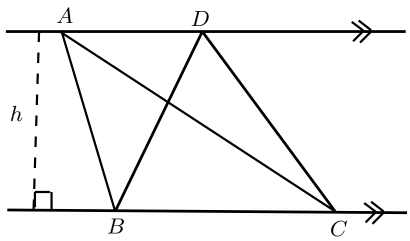
$$h_1 = h_2$$

$$\therefore \frac{\text{area } \triangle ABC}{\text{area } \triangle DEF} = \frac{\frac{1}{2} BC \times h_1}{\frac{1}{2} EF \times h_2} = \frac{BC}{EF}$$



- A special case of this happens when the bases of the triangles are equal:
  Triangles with equal bases between the same parallel lines have the same area.

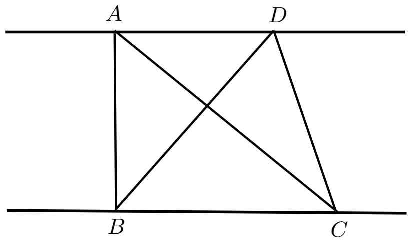
$$\text{area } \triangle ABC = \frac{1}{2} \, . \, h \, . \, BC = \text{ area } \triangle DBC$$



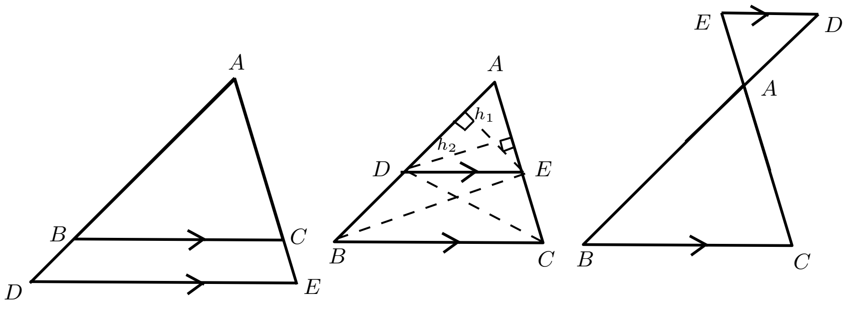
- Triangles on the same side of the same base, with equal areas, lie between parallel lines.

If area $\triangle ABC =$ area $\triangle BDC$

then $AD \parallel BC$



Theorem 1. *Proportion Theorem: A line drawn parallel to one side of a triangle divides the other two sides proportionally.*



Given:$\triangle ABC$ with line $DE \parallel BC$

R.T.P.:

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Proof: Draw h_1 from E perpendicular to AD, and h_2 from D perpendicular to AE.

Draw BE and CD.

$$\begin{aligned}
\frac{\text{area } \triangle ADE}{\text{area } \triangle BDE} &= \frac{\frac{1}{2}AD\,.\,h_1}{\frac{1}{2}DB\,.\,h_1} = \frac{AD}{DB} \\
\frac{\text{area } \triangle ADE}{\text{area } \triangle CED} &= \frac{\frac{1}{2}AE\,.\,h_2}{\frac{1}{2}EC\,.\,h_2} = \frac{AE}{EC} \\
\text{but area } \triangle BDE &= \text{area } \triangle CED \text{ (equal base and height)} \\
\therefore \frac{\text{area } \triangle ADE}{\text{area } \triangle BDE} &= \frac{\text{area } \triangle ADE}{\text{area } \triangle CED} \\
\therefore \frac{AD}{DB} &= \frac{AE}{EC}
\end{aligned}$$

$\therefore DE$ divides AB and AC proportionally.

Similarly,

$$\begin{aligned}
\frac{AD}{AB} &= \frac{AE}{AC} \\
\frac{AB}{BD} &= \frac{AC}{CE}
\end{aligned}$$

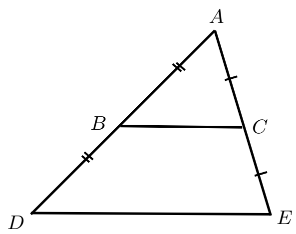
Following from Theorem 1, we can prove the midpoint theorem.

Theorem 2. *Midpoint Theorem: A line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side.*

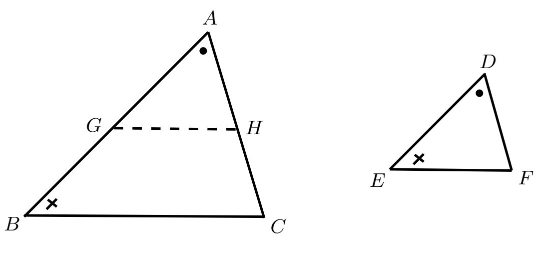
Proof:

This is a special case of the Proportionality Theorem (Theorem 1).

If $AB = BD$ and $AC = AE$,
and
$AD = AB + BD = 2AB$
$AE = AC + CB = 2AC$
then $DE \parallel BC$ and $BC = 2DE$.



Theorem 3. *Similarity Theorem 1: Equiangular triangles have their sides in proportion and are therefore similar.*



Given:$\triangle ABC$ and $\triangle DEF$ with $\hat{A} = \hat{D}$; $\hat{B} = \hat{E}$; $\hat{C} = \hat{F}$

R.T.P.:

$$\frac{AB}{DE} = \frac{AC}{DF}$$

Construct: G on AB, so that $AG = DE$
H on AC, so that $AH = DF$

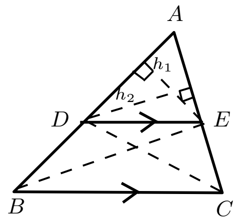
Proof: In $\triangle$'s AGH and DEF

$$\begin{array}{lll} & AG = DE; AH = DF & \text{(constant)} \\ & \hat{A} = \hat{D} & \text{(given)} \\ \therefore & \triangle AGH \equiv \triangle DEF & \text{(SAS)} \\ \therefore & A\hat{G}H = \hat{E} = \hat{B} & \\ \therefore & GH \parallel BC & \text{(corresponding } \angle\text{'s equal)} \\ \therefore & \frac{AG}{AB} = \frac{AH}{AC} & \text{(proportion theorem)} \\ \therefore & \frac{DE}{AB} = \frac{DF}{AC} & (AG = DE;\ AH = DF) \\ \therefore & \triangle ABC \,|||\, \triangle DEF & \end{array}$$

Tip

||| means "is similar to"

Theorem 4. *Similarity Theorem 2: Triangles with sides in proportion are equiangular and therefore similar.*



Given: $\triangle ABC$ with line DE such that

$$\frac{AD}{DB} = \frac{AE}{EC}$$

R.T.P.: $DE \parallel BC$; $\triangle ADE \,|||\, \triangle ABC$

Proof:

Draw h_1 from E perpendicular to AD, and h_2 from D perpendicular to AE.
Draw BE and CD.

$$\begin{array}{rcl} \frac{\text{area } \triangle ADE}{\text{area } \triangle BDE} & = & \frac{\frac{1}{2}AD\,.\,h_1}{\frac{1}{2}DB\,.\,h_1} = \frac{AD}{DB} \\ \frac{\text{area } \triangle ADE}{\text{area } \triangle CED} & = & \frac{\frac{1}{2}AE\,.\,h_2}{\frac{1}{2}EC\,.\,h_2} = \frac{AE}{EC} \\ \text{but } \frac{AD}{DB} & = & \frac{AE}{EC} \text{ (given)} \\ \therefore \frac{\text{area } \triangle ADE}{\text{area } \triangle BDE} & = & \frac{\text{area } \triangle ADE}{\text{area } \triangle CED} \\ \therefore \text{area } \triangle BDE & = & \text{area } \triangle CED \end{array}$$

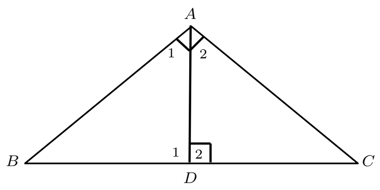
$$\begin{array}{rcl} \therefore DE \parallel BC & & \text{(same side of equal base DE, same area)} \\ \therefore A\hat{D}E & = & A\hat{B}C \text{ (corresponding } \angle\text{'s)} \\ \text{and } A\hat{E}D & = & A\hat{C}B \end{array}$$

$$\therefore \triangle ADE \text{ and } \triangle ABC \text{ are equiangular}$$

$$\therefore \triangle ADE \,|||\, \triangle ABC \text{ (AAA)}$$

Theorem 5. *Pythagoras' Theorem: The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.*

Given:$\triangle ABC$ with $\hat{A} = 90°$



R.T.P.: $BC^2 = AB^2 + AC^2$

Proof:

$$\begin{aligned} \text{Let } \hat{C} &= x \\ \therefore D\hat{A}C &= 90° - x \text{ (}\angle\text{'s of a }\triangle\text{)} \\ \therefore D\hat{A}B &= x \\ A\hat{B}D &= 90° - x \text{ (}\angle\text{'s of a }\triangle\text{)} \\ B\hat{D}A &= C\hat{D}A = \hat{A} = 90° \end{aligned}$$

$$\therefore \triangle ABD|||\triangle CBA \text{ and } \triangle CAD|||\triangle CBA \text{ (AAA)}$$

$$\therefore \frac{AB}{CB} = \frac{BD}{BA} = \left(\frac{AD}{CA}\right) \text{ and } \frac{CA}{CB} = \frac{CD}{CA} = \left(\frac{AD}{BA}\right)$$

$$\therefore AB^2 = CB \times BD \text{ and } AC^2 = CB \times CD$$

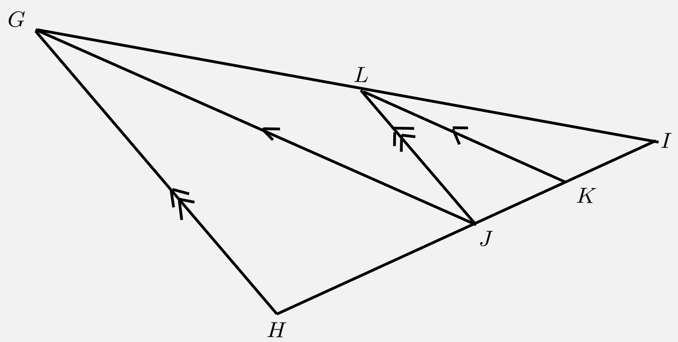
$$\begin{aligned} \therefore AB^2 + AC^2 &= CB(BD + CD) \\ &= CB(CB) \\ &= CB^2 \\ \text{i.e. } BC^2 &= AB^2 + AC^2 \end{aligned}$$

Example 4: *Triangle Geometry 1*

QUESTION

In $\triangle GHI$, $GH \parallel LJ$; $GJ \parallel LK$ *and* $\frac{JK}{KI} = \frac{5}{3}$. *Determine* $\frac{HJ}{KI}$.

SOLUTION



Step 1 : ***Identify similar triangles***

$$\begin{aligned} L\hat{I}J &= G\hat{I}H \\ J\hat{L}I &= H\hat{G}I && \text{(Corresponding } \angle\text{'s)} \\ \therefore \triangle LIJ &\,|||\, \triangle GIH && \text{(Equiangular } \triangle\text{'s)} \end{aligned}$$

$$\begin{aligned} L\hat{I}K &= G\hat{I}J \\ K\hat{L}I &= J\hat{G}I && \text{(Corresponding } \angle\text{'s)} \\ \therefore \triangle LIK &\,|||\, \triangle GIJ && \text{(Equiangular } \triangle\text{'s)} \end{aligned}$$

Step 2 : ***Use proportional sides***

$$\begin{aligned} \frac{HJ}{JI} &= \frac{GL}{LI} && (\triangle LIJ \,|||\, \triangle GIH) \\ \text{and } \frac{GL}{LI} &= \frac{JK}{KI} && (\triangle LIK \,|||\, \triangle GIJ) \\ &= \frac{5}{3} \\ \therefore \frac{HJ}{JI} &= \frac{5}{3} \end{aligned}$$

Step 3 : ***Rearrange to find the required ratio***

$$\frac{HJ}{KI} = \frac{HJ}{JI} \times \frac{JI}{KI}$$

We need to calculate $\frac{JI}{KI}$: We were given $\frac{JK}{KI} = \frac{5}{3}$ So rearranging, we have $JK = \frac{5}{3}KI$ And:

$$\begin{aligned} JI &= JK + KI \\ &= \frac{5}{3}KI + KI \\ &= \frac{8}{3}KI \\ \frac{JI}{KI} &= \frac{8}{3} \end{aligned}$$

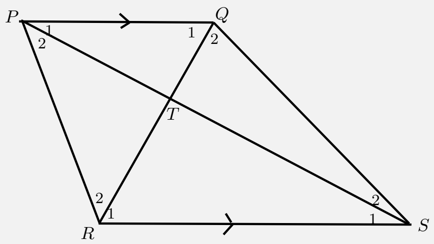
Using this relation:

$$\begin{aligned} &= \frac{5}{3} \times \frac{8}{3} \\ &= \frac{40}{9} \end{aligned}$$

Example 5: *Triangle Geometry 2*

QUESTION

$PQRS$ is a trapezium, with $PQ \parallel RS$.
Prove that $PT \,.\, TR = ST \,.\, TQ$.



SOLUTION

Step 4 : ***Identify similar triangles***

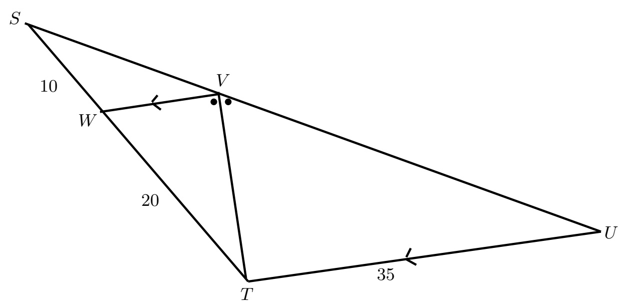
$$\begin{aligned} \hat{P}_1 &= \hat{S}_1 && \text{(alternate } \angle\text{'s)} \\ \hat{Q}_1 &= \hat{R}_1 && \text{(alternate } \angle\text{'s)} \\ \therefore \triangle PTQ &\,|||\, \triangle STR && \text{(equiangular } \triangle\text{'s)} \end{aligned}$$

Step 5 : ***Use proportional sides***

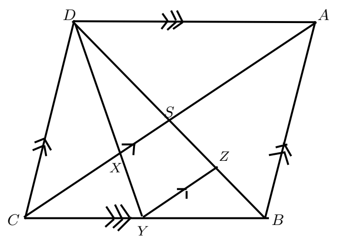
$$\begin{aligned} \frac{PT}{TQ} &= \frac{ST}{TR} && (\triangle PTQ|||\triangle STR) \\ \therefore PT \,.\, TR &= ST \,.\, TQ \end{aligned}$$

Exercise 16 - 2

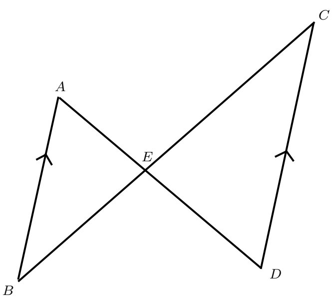
1. Calculate SV



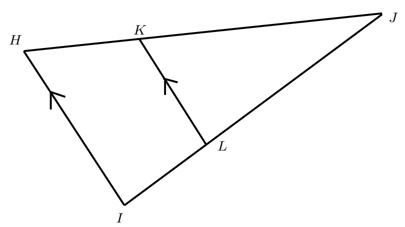
2. $\frac{CB}{YB} = \frac{3}{2}$. Find $\frac{DS}{SB}$.



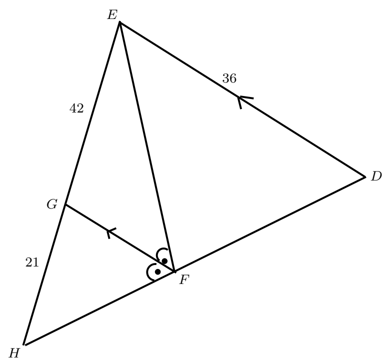
3. Given the following figure with the following lengths, find AE, EC and BE.
   $BC = 15$ cm, $AB = 4$ cm, $CD = 18$ cm, and $ED = 9$ cm.



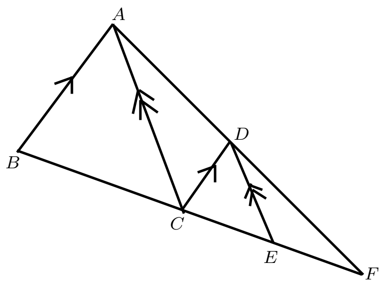
4. Using the following figure and lengths, find IJ and KJ.
   $HI = 26$ m, $KL = 13$ m, $JL = 9$ m and $HJ = 32$ m.



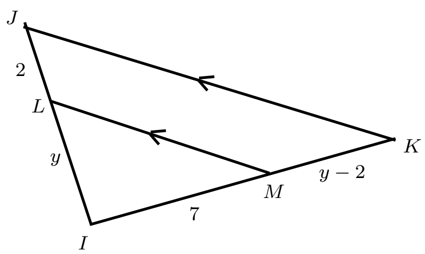
5. Find FH in the following figure.



6. $BF = 25$ m, $AB = 13$ m, $AD = 9$ m, $DF = 18$ m.
Calculate the lengths of BC, CF, CD, CE and EF, and find the ratio $\frac{DE}{AC}$.



7. If $LM \parallel JK$, calculate y.



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(1.) 012x (2.) 012y (3.) 012z (4.) 0130 (5.) 0131 (6.) 0132
(7.) 0133

16.5 Co-ordinate Geometry

EMBCK

Equation of a Line Between Two Points

EMBCL

See video: VMftf at www.everythingmaths.co.za

There are many different methods of specifying the requirements for determining the equation of a straight line. One option is to find the equation of a straight line, when two points are given.

Assume that the two points are $(x_1; y_1)$ and $(x_2; y_2)$, and we know that the general form of the equation for a straight line is:

$$\boxed{y = mx + c} \quad (16.1)$$

Tip

If you are asked to calculate the equation of a line passing through two points, use:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

to calculate m and then use:

$$y - y_1 = m(x - x_1)$$

to determine the equation.

So, to determine the equation of the line passing through our two points, we need to determine values for m (the gradient of the line) and c (the y-intercept of the line). The resulting equation is

$$\boxed{y - y_1 = m(x - x_1)} \quad (16.2)$$

where $(x_1; y_1)$ are the co-ordinates of either given point.

Extension:	*Finding the second equation for a straight line*

This is an example of a set of simultaneous equations, because we can write:

$$\begin{aligned} y_1 &= mx_1 + c & (16.3) \\ y_2 &= mx_2 + c & (16.4) \end{aligned}$$

We now have two equations, with two unknowns, m and c.

$$\begin{aligned} \text{Subtract (16.3) from (16.4)} \quad y_2 - y_1 &= mx_2 - mx_1 & (16.5) \\ \therefore \quad m &= \frac{y_2 - y_1}{x_2 - x_1} & (16.6) \\ \text{Re-arrange (16.3) to obtain } c \quad y_1 &= mx_1 + c & (16.7) \\ c &= y_1 - mx_1 & (16.8) \end{aligned}$$

Now, to make things a bit easier to remember, substitute (16.7) into (16.1):

$$\begin{aligned} y &= mx + c & (16.9) \\ &= mx + (y_1 - mx_1) & (16.10) \\ \text{which can be re-arranged to:} \quad y - y_1 &= m(x - x_1) & (16.11) \end{aligned}$$

For example, the equation of the straight line passing through $(-1; 1)$ and $(2; 2)$ is given by first calculating m

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 1}{2 - (-1)} \\ &= \frac{1}{3} \end{aligned}$$

and then substituting this value into

$$y - y_1 = m(x - x_1)$$

to obtain

$$y - y_1 = \frac{1}{3}(x - x_1).$$

Then substitute $(-1; 1)$ to obtain

$$\begin{aligned} y - (1) &= \frac{1}{3}(x - (-1)) \\ y - 1 &= \frac{1}{3}x + \frac{1}{3} \\ y &= \frac{1}{3}x + \frac{1}{3} + 1 \\ y &= \frac{1}{3}x + \frac{4}{3} \end{aligned}$$

So, $y = \frac{1}{3}x + \frac{4}{3}$ passes through $(-1; 1)$ and $(2; 2)$.

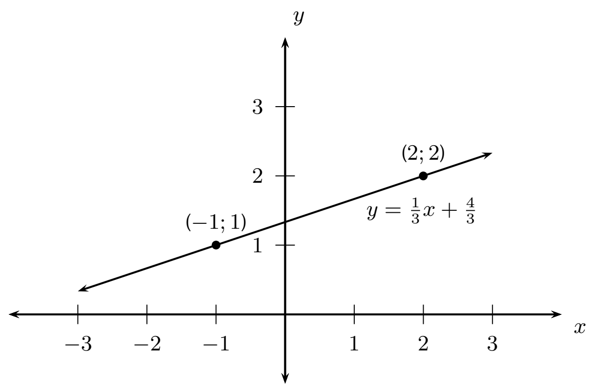

Figure 16.2: The equation of the line passing through $(-1; 1)$ and $(2; 2)$ is $y = \frac{1}{3}x + \frac{4}{3}$.

Example 6: *Equation of Straight Line*

QUESTION

Find the equation of the straight line passing through $(-3; 2)$ *and* $(5; 8)$.

SOLUTION

Step 1 : ***Label the points***

$$\begin{aligned}(x_1; y_1) &= (-3; 2)\\ (x_2; y_2) &= (5; 8)\end{aligned}$$

Step 2 : ***Calculate the gradient***

$$\begin{aligned}m &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{8 - 2}{5 - (-3)}\\ &= \frac{6}{5 + 3}\\ &= \frac{6}{8}\\ &= \frac{3}{4}\end{aligned}$$

Step 3 : ***Determine the equation of the line***

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - (2) &= \frac{3}{4}(x - (-3))\\ y &= \frac{3}{4}(x + 3) + 2\\ &= \frac{3}{4}x + \frac{3}{4}.3 + 2\\ &= \frac{3}{4}x + \frac{9}{4} + \frac{8}{4}\\ &= \frac{3}{4}x + \frac{17}{4}\end{aligned}$$

Step 4 : ***Write the final answer***

The equation of the straight line that passes through $(-3; 2)$ and $(5; 8)$ is $y = \frac{3}{4}x + \frac{17}{4}$.

Equation of a Line Through One Point and Parallel or Perpendicular to Another Line

EMBCM

Another method of determining the equation of a straight-line is to be given one point, $(x_1; y_1)$, and to be told that the line is parallel or perpendicular to another line. If the equation of the unknown line is

$y = mx + c$ and the equation of the second line is $y = m_0x + c_0$, then we know the following:

$$\text{If the lines are parallel, then} \quad m = m_0 \tag{16.12}$$
$$\text{If the lines are perpendicular, then} \quad m \times m_0 = -1 \tag{16.13}$$

Once we have determined a value for m, we can then use the given point together with:

$$y - y_1 = m(x - x_1)$$

to determine the equation of the line.

For example, find the equation of the line that is parallel to $y = 2x - 1$ and that passes through $(-1; 1)$.

First we determine m, the slope of the line we are trying to find. Since the line we are looking for is parallel to $y = 2x - 1$,

$$m = 2$$

The equation is found by substituting m and $(-1; 1)$ into:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - (-1)) \\ y - 1 &= 2(x + 1) \\ y - 1 &= 2x + 2 \\ y &= 2x + 2 + 1 \\ y &= 2x + 3 \end{aligned}$$

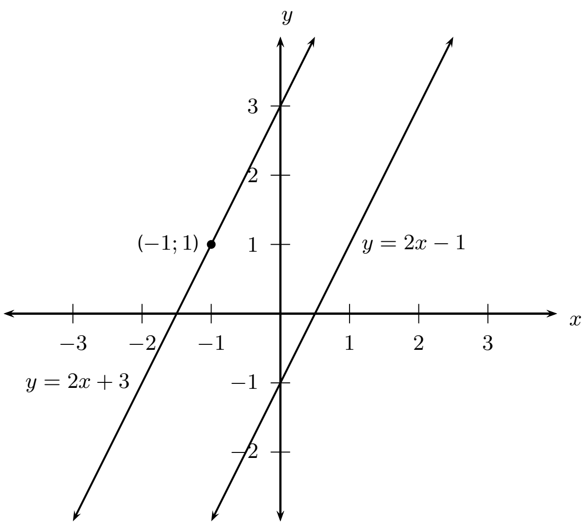

Figure 16.3: The equation of the line passing through $(-1; 1)$ and parallel to $y = 2x-1$ is $y = 2x+3$. It can be seen that the lines are parallel to each other. You can test this by using your ruler and measuring the perpendicular distance between the lines at different points.

Inclination of a Line

EMBCN

In Figure 16.4(a), we see that the line makes an angle θ with the x-axis. This angle is known as the *inclination* of the line and it is sometimes interesting to know what the value of θ is.

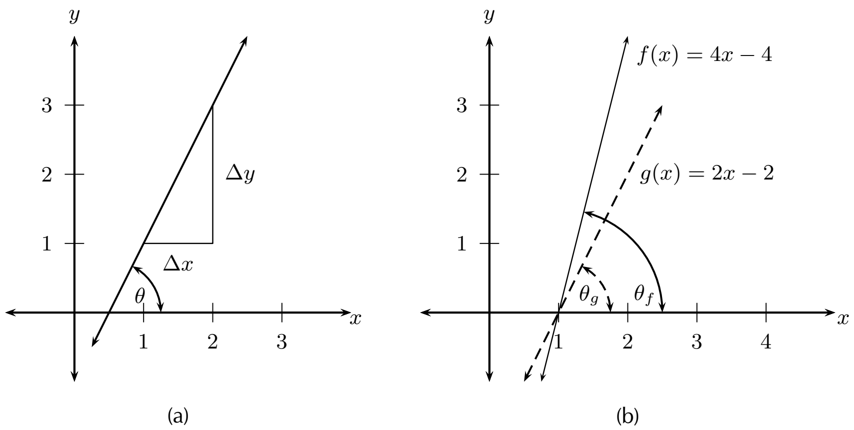

Figure 16.4: (a) A line makes an angle θ with the x-axis. (b) The angle is dependent on the gradient. If the gradient of f is m_f and the gradient of g is m_g then $m_f > m_g$ and $\theta_f > \theta_g$.

Firstly, we note that if the gradient changes, then the value of θ changes (Figure 16.4(b)), so we suspect that the inclination of a line is related to the gradient. We know that the gradient is a ratio of a change in the y-direction to a change in the x-direction.

$$m = \frac{\Delta y}{\Delta x}$$

But, in Figure 16.4(a) we see that

$$\begin{aligned} \tan\theta &= \frac{\Delta y}{\Delta x} \\ \therefore m &= \tan\theta \end{aligned}$$

For example, to find the inclination of the line $y = x$, we know $m = 1$

$$\begin{aligned} \therefore \tan\theta &= 1 \\ \therefore \theta &= 45^\circ \end{aligned}$$

Exercise 16 - 3

1. Find the equations of the following lines
   (a) through points $(-1; 3)$ and $(1; 4)$
   (b) through points $(7; -3)$ and $(0; 4)$
   (c) parallel to $y = \frac{1}{2}x + 3$ passing through $(-1; 3)$
   (d) perpendicular to $y = -\frac{1}{2}x + 3$ passing through $(-1; 2)$
   (e) perpendicular to $2y + x = 6$ passing through the origin
2. Find the inclination of the following lines
   (a) $y = 2x - 3$
   (b) $y = \frac{1}{3}x - 7$
   (c) $4y = 3x + 8$
   (d) $y = -\frac{2}{3}x + 3$ (Hint: if m is negative θ must be in the second quadrant)
   (e) $3y + x - 3 = 0$
3. Show that the line $y = k$ for any constant k is parallel to the x-axis. (Hint: Show that the inclination of this line is 0°.)

4. Show that the line $x = k$ for any constant k is parallel to the y-axis. (Hint: Show that the inclination of this line is $90°$.)

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(1.) 0134 (2.) 0135 (3.) 0136 (4.) 0137

16.6 Transformations

EMBCO

Rotation of a Point

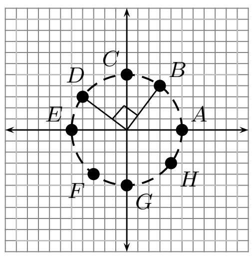
EMBCP

When something is moved around a fixed point, we say that it is *rotated* about the point. What happens to the coordinates of a point that is rotated by $90°$ or $180°$ around the origin?

Activity: *Rotation of a Point by* $90°$

Complete the table, by filling in the coordinates of the points shown in the figure.

Point	**x-coordinate**	**y-coordinate**
A		
B		
C		
D		
E		
F		
G		
H		

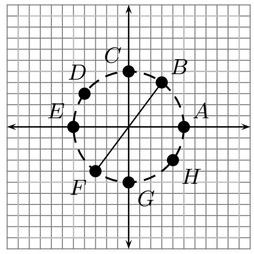


What do you notice about the x-coordinates? What do you notice about the y-coordinates? What would happen to the coordinates of point A, if it was rotated to the position of point C? What about if point B rotated to the position of D?

Activity: *Rotation of a Point by* $180°$

Complete the table, by filling in the coordinates of the points shown in the figure.

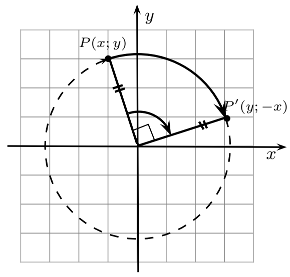
Point	x-coordinate	y-coordinate
A		
B		
C		
D		
E		
F		
G		
H		



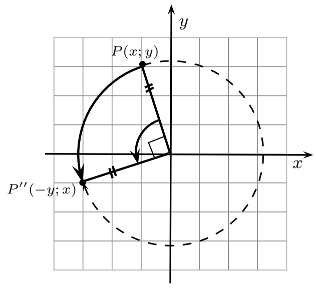
What do you notice about the x-coordinates? What do you notice about the y-coordinates? What would happen to the coordinates of point A, if it was rotated to the position of point E? What about point F rotated to the position of B?

From these activities you should have come to the following conclusions:

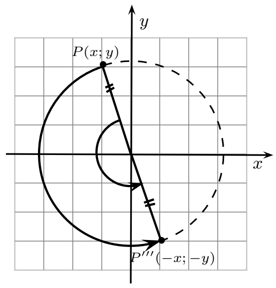
- $90°$ clockwise rotation:
  The image of a point $P(x;y)$ rotated clockwise through $90°$ around the origin is $P'(y;-x)$.
  We write the rotation as $(x;y) \to (y;-x)$.



- $90°$ anticlockwise rotation:
  The image of a point $P(x;y)$ rotated anticlockwise through $90°$ around the origin is $P'(-y;x)$.
  We write the rotation as $(x;y) \to (-y;x)$.



- $180°$ rotation:
  The image of a point $P(x;y)$ rotated through $180°$ around the origin is $P'(-x;-y)$.
  We write the rotation as $(x;y) \to (-x;-y)$.



Exercise 16 - 4

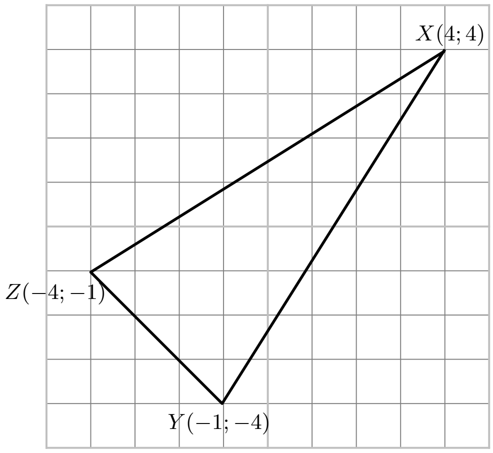
1. For each of the following rotations about the origin:
   (i) Write down the rule.
   (ii) Draw a diagram showing the direction of rotation.
   (a) OA is rotated to OA' with $A(4;2)$ and $A'(-2;4)$
   (b) OB is rotated to OB' with $B(-2;5)$ and $B'(5;2)$

(c) OC is rotated to OC' with $C(-1;-4)$ and $C'(1;4)$

2. Copy ΔXYZ onto squared paper. The co-ordinates are given on the picture.

   (a) Rotate ΔXYZ anti-clockwise through an angle of $90°$ about the origin to give $\Delta X'Y'Z'$. Give the co-ordinates of X', Y' and Z'.

   (b) Rotate ΔXYZ through $180°$ about the origin to give $\Delta X''Y''Z''$. Give the co-ordinates of X'', $Y''$and Z''.



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(1.) 0138 (2.) 0139

Enlargement of a Polygon

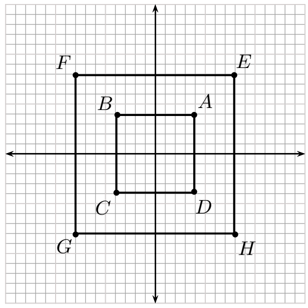
EMBCQ

When something is made larger, we say that it is *enlarged*. What happens to the coordinates of a polygon that is enlarged by a factor k?

Activity: *Enlargement of a Polygon*

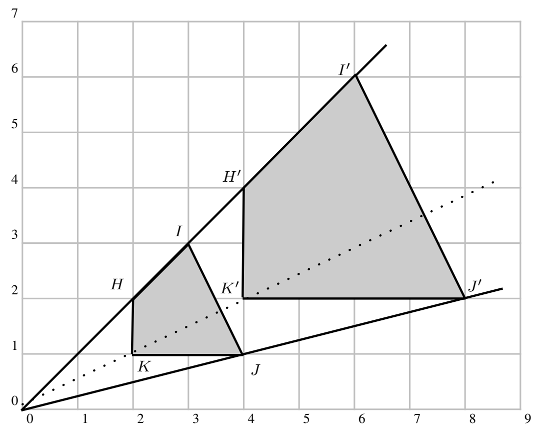
Complete the table, by filling in the coordinates of the points shown in the figure. Assume each small square on the plot is 1 unit.

Point	x-coordinate	y-coordinate
A		
B		
C		
D		
E		
F		
G		
H		



What do you notice about the x-coordinates? What do you notice about the y-coordinates? What would happen to the coordinates of point A, if the square $ABCD$ was enlarged by a factor of 2?

Activity: *Enlargement of a Polygon*



In the figure quadrilateral $HIJK$ has been enlarged by a factor of 2 through the origin to become $H'I'J'K'$. Complete the following table using the information in the figure.

Co-ordinate	Co-ordinate′	Length	Length′
H = (;)	H' = (;)	OH =	OH' =
I = (;)	I' = (;)	OI =	OI' =
J = (;)	J' = (;)	OJ =	OJ' =
K = (;)	K' = (;)	OK =	OK' =

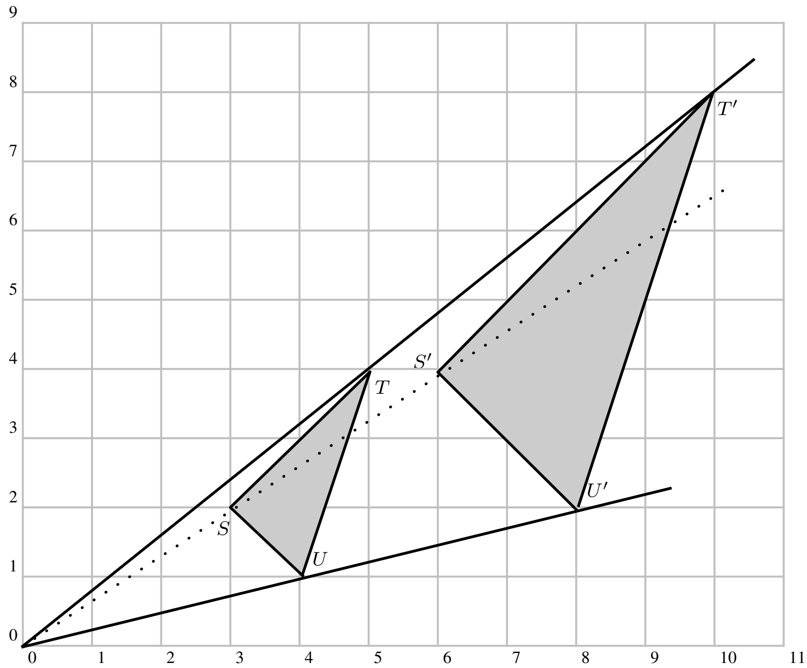
What conclusions can you draw about

1. the co-ordinates
2. the lengths when we enlarge by a factor of 2?

We conclude as follows:
Let the vertices of a triangle have co-ordinates $S(x_1; y_1)$, $T(x_2; y_2)$, $U(x_3; y_3)$. $\triangle S'T'U'$ is an enlargement through the origin of $\triangle STU$ by a factor of c ($c > 0$).

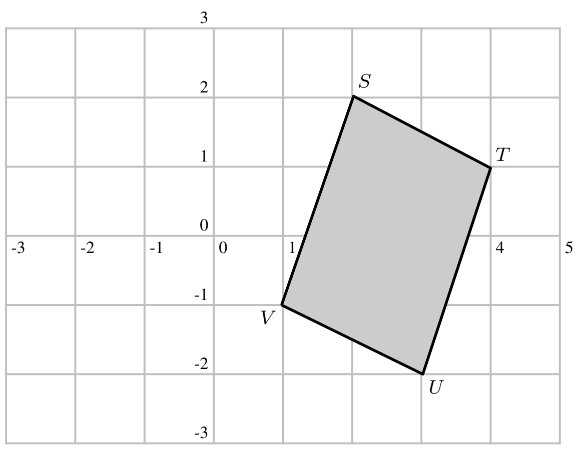
- $\triangle STU$ is a reduction of $\triangle S'T'U'$ by a factor of c.
- $\triangle S'T'U'$ can alternatively be seen as an reduction through the origin of $\triangle STU$ by a factor of $\frac{1}{c}$. (Note that a reduction by $\frac{1}{c}$ is the same as an enlargement by c).
- The vertices of $\triangle S'T'U'$ are $S'(cx_1; cy_1)$, $T'(cx_2,cy_2)$, $U'(cx_3,cy_3)$.

- The distances from the origin are $OS' = (c\,.\,OS)$, $OT' = (c\,.\,OT)$ and $OU' = (c\,.\,OU)$.



Chapter 16 End of Chapter Exercises

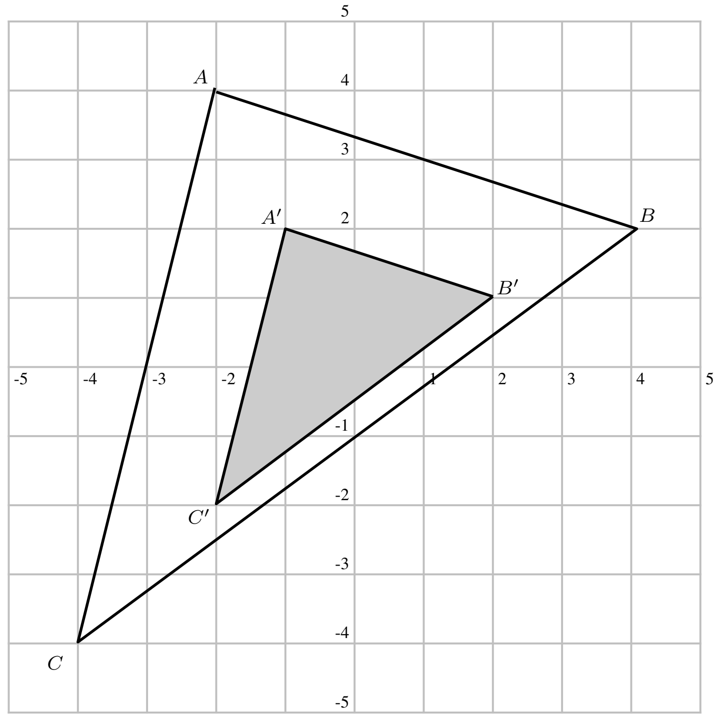
1. Copy polygon $STUV$ onto squared paper and then answer the following questions.



   (a) What are the co-ordinates of polygon $STUV$?
   (b) Enlarge the polygon through the origin by a constant factor of $c = 2$. Draw this on the same grid. Label it $S'T'U'V'$.

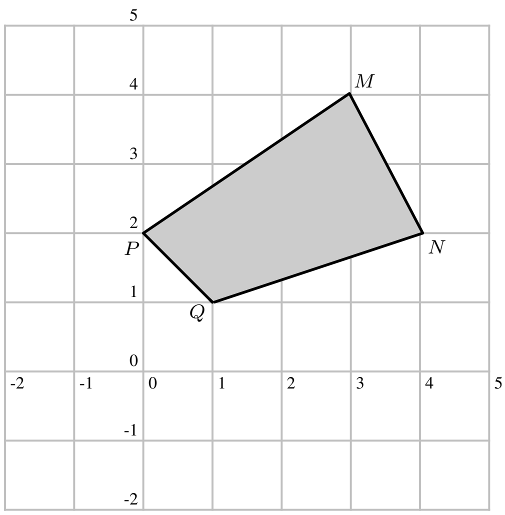
(c) What are the co-ordinates of the vertices of $S'T'U'V'$?

2. $\triangle ABC$ is an enlargement of $\triangle A'B'C'$ by a constant factor of k through the origin.



(a) What are the co-ordinates of the vertices of $\triangle ABC$ and $\triangle A'B'C'$?
(b) Giving reasons, calculate the value of k.
(c) If the area of $\triangle ABC$ is m times the area of $\triangle A'B'C'$, what is m?

3. Examine the polygon below.



(a) What are the co-ordinates of the vertices of polygon $MNPQ$?
(b) Enlarge the polygon through the origin by using a constant factor of $c = 3$, obtaining polygon $M'N'P'Q'$. Draw this on the same set of axes.

(c) What are the co-ordinates of the new vertices?

(d) Now draw $M''N''P''Q''$ which is an anticlockwise rotation of $MNPQ$ by $90°$ around the origin.

(e) Find the inclination of OM''.

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(1.) 01zi (2.) 01zj (3.) 01zk

Trigonometry 17

17.1 Introduction EMBCR

Building on Grade 10 Trigonometry, we will look at more general forms of the the basic trigonometric functions next. We will use graphs and algebra to analyse the properties of these functions. We will also see that different trigonometric functions are closely related through a number of mathematical identities.

See introductory video: VMfva at www.everythingmaths.co.za

17.2 Graphs of Trigonometric Functions EMBCS

Functions of the Form $y = \sin(k\theta)$ EMBCT

In the equation, $y = \sin(k\theta)$, k is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 17.1 for the function $f(\theta) = \sin(2\theta)$.

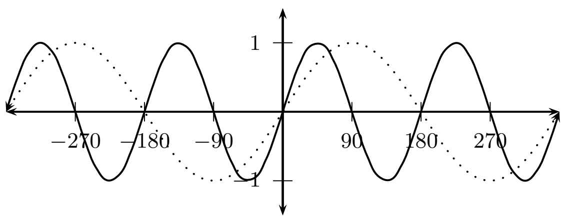

Figure 17.1: Graph of $f(\theta) = \sin(2\theta)$ (solid line) and the graph of $g(\theta) = \sin(\theta)$ (dotted line).

Exercise 17 - 1

On the same set of axes, plot the following graphs:

1. $a(\theta) = \sin 0{,}5\theta$
2. $b(\theta) = \sin 1\theta$
3. $c(\theta) = \sin 1{,}5\theta$

4. $d(\theta) = \sin 2\theta$

5. $e(\theta) = \sin 2{,}5\theta$

Use your results to deduce the effect of k.

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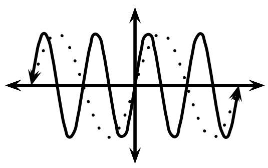
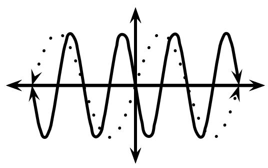
(1.) 013c

You should have found that the value of k affects the period or frequency of the graph. Notice that in the case of the sine graph, the period (length of one wave) is given by $\frac{360^\circ}{k}$.

These different properties are summarised in Table 17.1.

Table 17.1: Table summarising general shapes and positions of graphs of functions of the form $y = \sin(kx)$. The curve $y = \sin(x)$ is shown as a dotted line.

$k > 0$	$k < 0$
	

Domain and Range

For $f(\theta) = \sin(k\theta)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \sin(k\theta)$ is $\{f(\theta) : f(\theta) \in [-1; 1]\}$.

Intercepts

For functions of the form, $y = \sin(k\theta)$, the details of calculating the intercepts with the y axis are given.

There are many x-intercepts.

The y-intercept is calculated by setting $\theta = 0$:

$$\begin{aligned} y &= \sin(k\theta) \\ y_{int} &= \sin(0) \\ &= 0 \end{aligned}$$

Functions of the Form $y = \cos(k\theta)$ EMBCU

In the equation, $y = \cos(k\theta)$, k is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 17.2 for the function $f(\theta) = \cos(2\theta)$.

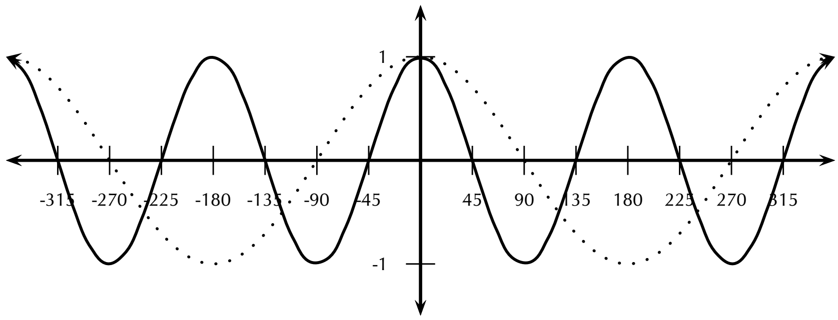


Figure 17.2: Graph of $f(\theta) = \cos(2\theta)$ (solid line) and the graph of $g(\theta) = cos(\theta)$ (dotted line).

Exercise 17 - 2

On the same set of axes, plot the following graphs:

1. $a(\theta) = \cos 0{,}5\theta$
2. $b(\theta) = \cos 1\theta$
3. $c(\theta) = \cos 1{,}5\theta$
4. $d(\theta) = \cos 2\theta$
5. $e(\theta) = \cos 2{,}5\theta$

Use your results to deduce the effect of k.

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(1.) 013h

You should have found that the value of k affects the period or frequency of the graph. The period of the cosine graph is given by $\frac{360°}{k}$.

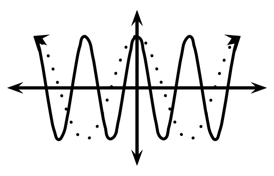
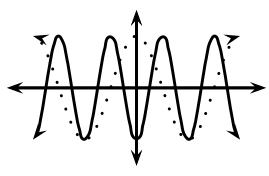
These different properties are summarised in Table 17.2.

Domain and Range

For $f(\theta) = \cos(k\theta)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \cos(k\theta)$ is $\{f(\theta) : f(\theta) \in [-1; 1]\}$.

Table 17.2: Table summarising general shapes and positions of graphs of functions of the form $y = \cos(kx)$. The curve $y = \cos(x)$ is plotted with a dotted line.

$k > 0$	$k < 0$
	

Intercepts

For functions of the form, $y = \cos(k\theta)$, the details of calculating the intercepts with the y axis are given.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= \cos(k\theta) \\ y_{int} &= \cos(0) \\ &= 1 \end{aligned}$$

Functions of the Form $y = \tan(k\theta)$ EMBCV

In the equation, $y = \tan(k\theta)$, k is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 17.3 for the function $f(\theta) = \tan(2\theta)$.

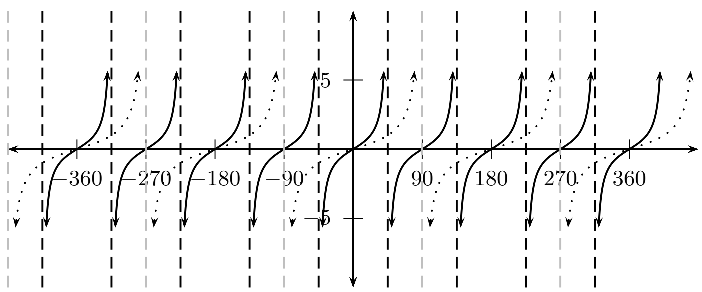


Figure 17.3: The graph of $f(\theta) = \tan(2\theta)$ (solid line) and the graph of $g(\theta) = \tan(\theta)$ (dotted line). The asymptotes are shown as dashed lines.

Exercise 17 - 3

On the same set of axes, plot the following graphs:

1. $a(\theta) = \tan 0{,}5\theta$

2. $b(\theta) = \tan 1\theta$
3. $c(\theta) = \tan 1{,}5\theta$
4. $d(\theta) = \tan 2\theta$
5. $e(\theta) = \tan 2{,}5\theta$

Use your results to deduce the effect of k.

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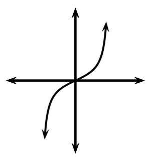
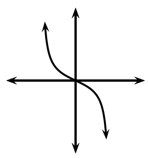
(1.) 013n

You should have found that, once again, the value of k affects the periodicity (i.e. frequency) of the graph. As k increases, the graph is more tightly packed. As k decreases, the graph is more spread out. The period of the tan graph is given by $\frac{180^\circ}{k}$.

These different properties are summarised in Table 17.3.

Table 17.3: Table summarising general shapes and positions of graphs of functions of the form $y = \tan(k\theta)$.

$k > 0$	$k < 0$
	

Domain and Range

For $f(\theta) = \tan(k\theta)$, the domain of one branch is $\{\theta : \theta \in (-\frac{90^\circ}{k}; \frac{90^\circ}{k})\}$ because the function is undefined for $\theta = -\frac{90^\circ}{k}$ and $\theta = \frac{90^\circ}{k}$.

The range of $f(\theta) = \tan(k\theta)$ is $\{f(\theta) : f(\theta) \in (-\infty; \infty)\}$.

Intercepts

For functions of the form, $y = \tan(k\theta)$, the details of calculating the intercepts with the x and y axis are given.

There are many x-intercepts; each one is halfway between the asymptotes.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= \tan(k\theta) \\ y_{int} &= \tan(0) \\ &= 0 \end{aligned}$$

Asymptotes

The graph of $\tan k\theta$ has asymptotes because as $k\theta$ approaches 90°, $\tan k\theta$ approaches infinity. In other words, there is no defined value of the function at the asymptote values.

Functions of the Form $y = \sin(\theta + p)$

EMBCW

In the equation, $y = \sin(\theta + p)$, p is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 17.4 for the function $f(\theta) = \sin(\theta + 30°)$.

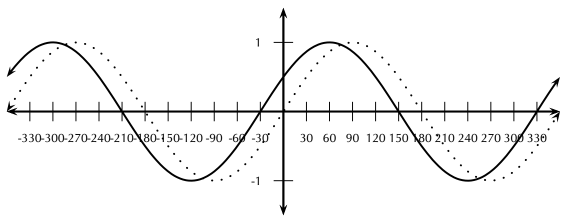


Figure 17.4: Graph of $f(\theta) = \sin(\theta + 30°)$ (solid line) and the graph of $g(\theta) = \sin(\theta)$ (dotted line).

Exercise 17 - 4

On the same set of axes, plot the following graphs:

1. $a(\theta) = \sin(\theta - 90°)$
2. $b(\theta) = \sin(\theta - 60°)$
3. $c(\theta) = \sin\theta$
4. $d(\theta) = \sin(\theta + 90°)$
5. $e(\theta) = \sin(\theta + 180°)$

Use your results to deduce the effect of p.

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(1.) 013t

You should have found that the value of p affects the position of the graph along the y-axis (i.e. the y-intercept) and the position of the graph along the x-axis (i.e. the *phase shift*). The p value shifts the graph horizontally. If p is positive, the graph shifts left and if p is negative the graph shifts right.

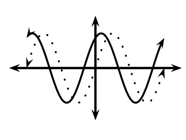
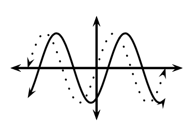
These different properties are summarised in Table 17.4.

Domain and Range

For $f(\theta) = \sin(\theta + p)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \sin(\theta + p)$ is $\{f(\theta) : f(\theta) \in [-1; 1]\}$.

Table 17.4: Table summarising general shapes and positions of graphs of functions of the form $y = \sin(\theta + p)$. The curve $y = \sin(\theta)$ is plotted with a dotted line.

$p > 0$	$p < 0$
	

Intercepts

For functions of the form, $y = \sin(\theta + p)$, the details of calculating the intercept with the y axis are given.

The y-intercept is calculated as follows: set $\theta = 0°$

$$\begin{aligned} y &= \sin(\theta + p) \\ y_{int} &= \sin(0 + p) \\ &= \sin(p) \end{aligned}$$

Functions of the Form $y = \cos(\theta + p)$

EMBCX

In the equation, $y = \cos(\theta + p)$, p is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 17.5 for the function $f(\theta) = \cos(\theta + 30°)$.

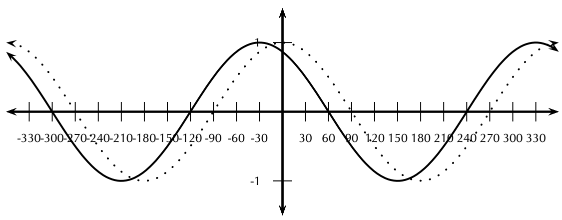


Figure 17.5: Graph of $f(\theta) = \cos(\theta + 30°)$ (solid line) and the graph of $g(\theta) = \cos(\theta)$ (dotted line).

Exercise 17 - 5

On the same set of axes, plot the following graphs:

1. $a(\theta) = \cos(\theta - 90°)$
2. $b(\theta) = \cos(\theta - 60°)$
3. $c(\theta) = \cos\theta$
4. $d(\theta) = \cos(\theta + 90°)$

5. $e(\theta) = \cos(\theta + 180°)$

Use your results to deduce the effect of p.

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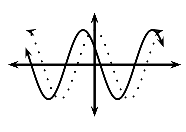
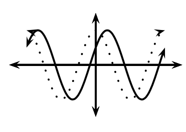
(1.) 013y

You should have found that the value of p affects the y-intercept and phase shift of the graph. As in the case of the sine graph, positive values of p shift the cosine graph left while negative p values shift the graph right.

These different properties are summarised in Table 17.5.

Table 17.5: Table summarising general shapes and positions of graphs of functions of the form $y = \cos(\theta + p)$. The curve $y = \cos\theta$ is plotted with a dotted line.

$p > 0$	$p < 0$
	

Domain and Range

For $f(\theta) = \cos(\theta + p)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \cos(\theta + p)$ is $\{f(\theta) : f(\theta) \in [-1; 1]\}$.

Intercepts

For functions of the form, $y = \cos(\theta + p)$, the details of calculating the intercept with the y axis are given.

The y-intercept is calculated as follows: set $\theta = 0°$

$$\begin{aligned} y &= \cos(\theta + p) \\ y_{int} &= \cos(0 + p) \\ &= \cos(p) \end{aligned}$$

Functions of the Form $y = \tan(\theta + p)$ EMBCY

In the equation, $y = \tan(\theta + p)$, p is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 17.6 for the function $f(\theta) = \tan(\theta + 30°)$.

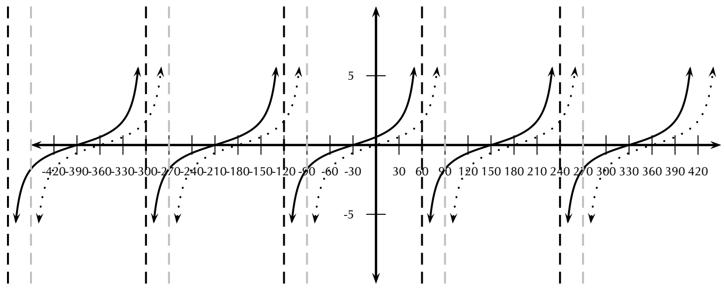


Figure 17.6: The graph of $f(\theta) = \tan(\theta + 30°)$ (solid lines) and the graph of $g(\theta) = \tan(\theta)$ (dotted lines).

Exercise 17 - 6

On the same set of axes, plot the following graphs:

1. $a(\theta) = \tan(\theta - 90°)$
2. $b(\theta) = \tan(\theta - 60°)$
3. $c(\theta) = \tan\theta$
4. $d(\theta) = \tan(\theta + 60°)$
5. $e(\theta) = \tan(\theta + 180°)$

Use your results to deduce the effect of p.

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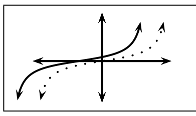
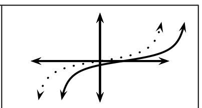
(1.) 0143

You should have found that the value of p once again affects the y-intercept and phase shift of the graph. There is a horizontal shift to the left if p is positive and to the right if p is negative.

These different properties are summarised in Table 17.6.

Table 17.6: Table summarising general shapes and positions of graphs of functions of the form $y = \tan(\theta + p)$. The curve $y = \tan(\theta)$ is plotted with a dotted line.

$k > 0$	$k < 0$
	

Domain and Range

For $f(\theta) = \tan(\theta + p)$, the domain for one branch is $\{\theta : \theta \in (-90° - p; 90° - p\}$ because the function is undefined for $\theta = -90° - p$ and $\theta = 90° - p$.

The range of $f(\theta) = \tan(\theta + p)$ is $\{f(\theta) : f(\theta) \in (-\infty; \infty)\}$.

Intercepts

For functions of the form, $y = \tan(\theta + p)$, the details of calculating the intercepts with the y axis are given.

The y-intercept is calculated as follows: set $\theta = 0°$

$$\begin{aligned} y &= \tan(\theta + p) \\ y_{int} &= \tan(p) \end{aligned}$$

Asymptotes

The graph of $\tan(\theta + p)$ has asymptotes because as $\theta + p$ approaches $90°$, $\tan(\theta + p)$ approaches infinity. Thus, there is no defined value of the function at the asymptote values.

Exercise 17 - 7

Using your knowledge of the effects of p and k draw a rough sketch of the following graphs without a table of values.

1. $y = \sin 3x$
2. $y = -\cos 2x$
3. $y = \tan \frac{1}{2}x$
4. $y = \sin(x - 45°)$
5. $y = \cos(x + 45°)$
6. $y = \tan(x - 45°)$
7. $y = 2 \sin 2x$
8. $y = \sin(x + 30°) + 1$

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(1.) 0148 (2.) 0149 (3.) 014a (4.) 014b (5.) 014c (6.) 014d
(7.) 014e (8.) 014f

17.3 Trigonometric Identities

EMBCZ

Deriving Values of Trigonometric Functions for $30°$, $45°$ and $60°$

EMBDA

Keeping in mind that trigonometric functions apply only to right-angled triangles, we can derive values of trigonometric functions for $30°$, $45°$ and $60°$. We shall start with $45°$ as this is the easiest.

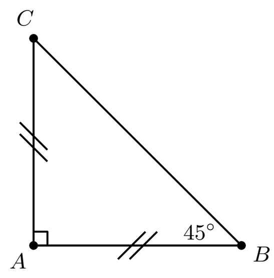


Figure 17.7: An isosceles right angled triangle.

Take any right-angled triangle with one angle $45°$. Then, because one angle is $90°$, the third angle is also $45°$. So we have an isosceles right-angled triangle as shown in Figure 17.7.

If the two equal sides are of length a, then the hypotenuse, h, can be calculated as:

$$\begin{aligned} h^2 &= a^2 + a^2 \\ &= 2a^2 \\ \therefore \quad h &= \sqrt{2}a \end{aligned}$$

So, we have:

$$\begin{aligned} \sin(45°) &= \frac{\text{opposite}(45°)}{\text{hypotenuse}} \\ &= \frac{a}{\sqrt{2}a} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \cos(45°) &= \frac{\text{adjacent}(45°)}{\text{hypotenuse}} \\ &= \frac{a}{\sqrt{2}a} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \tan(45°) &= \frac{\text{opposite}(45°)}{\text{adjacent}(45°)} \\ &= \frac{a}{a} \\ &= 1 \end{aligned}$$

We can try something similar for $30°$ and $60°$. We start with an equilateral triangle and we bisect one angle as shown in Figure 17.8. This gives us the right-angled triangle that we need, with one angle of $30°$ and one angle of $60°$.

If the equal sides are of length a, then the base is $\frac{1}{2}a$ and the length of the vertical side, v, can be calculated as:

$$\begin{aligned} v^2 &= a^2 - (\frac{1}{2}a)^2 \\ &= a^2 - \frac{1}{4}a^2 \\ &= \frac{3}{4}a^2 \\ \therefore \quad v &= \frac{\sqrt{3}}{2}a \end{aligned}$$

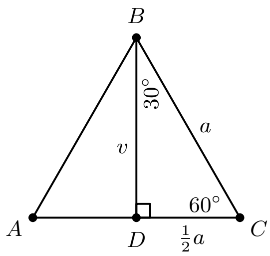


Figure 17.8: An equilateral triangle with one angle bisected.

So, we have:

$$\begin{aligned}\sin(30°) &= \frac{\text{opposite}(30°)}{\text{hypotenuse}} \\ &= \frac{\frac{a}{2}}{a} \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\cos(30°) &= \frac{\text{adjacent}(30°)}{\text{hypotenuse}} \\ &= \frac{\frac{\sqrt{3}}{2}a}{a} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\tan(30°) &= \frac{\text{opposite}(30°)}{\text{adjacent}(30°)} \\ &= \frac{\frac{a}{2}}{\frac{\sqrt{3}}{2}a} \\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

$$\begin{aligned}\sin(60°) &= \frac{\text{opposite}(60°)}{\text{hypotenuse}} \\ &= \frac{\frac{\sqrt{3}}{2}a}{a} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

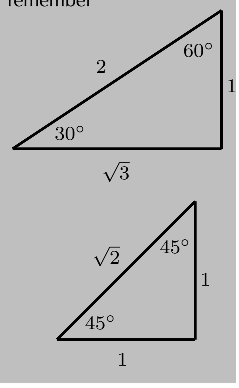
$$\begin{aligned}\cos(60°) &= \frac{\text{adjacent}(60°)}{\text{hypotenuse}} \\ &= \frac{\frac{a}{2}}{a} \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\tan(60°) &= \frac{\text{opposite}(60°)}{\text{adjacent}(60°)} \\ &= \frac{\frac{\sqrt{3}}{2}a}{\frac{a}{2}} \\ &= \sqrt{3}\end{aligned}$$

You do not have to memorise these identities if you know how to work them out.

Tip

Two useful triangles to remember



Alternate Definition for $\tan\theta$

EMBDB

We know that $\tan\theta$ is defined as:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

This can be written as:

$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \times \frac{\text{hypotenuse}}{\text{hypotenuse}} \\ &= \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}}\end{aligned}$$

But, we also know that $\sin\theta$ is defined as:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

and that $\cos\theta$ is defined as:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Therefore, we can write

$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}} \\ &= \sin\theta \times \frac{1}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta}\end{aligned}$$

Tip

$\tan\theta$ can also be defined as:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

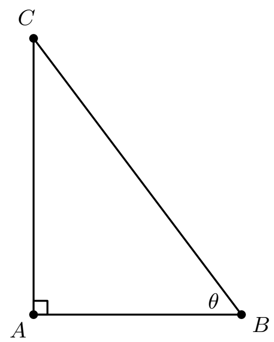
A Trigonometric Identity

EMBDC

One of the most useful results of the trigonometric functions is that they are related to each other. We have seen that $\tan\theta$ can be written in terms of $\sin\theta$ and $\cos\theta$. Similarly, we shall show that:

$$\sin^2\theta + \cos^2\theta = 1$$

We shall start by considering $\triangle ABC$,



We see that:

$$\sin\theta = \frac{AC}{BC}$$

and

$$\cos\theta = \frac{AB}{BC}.$$

We also know from the Theorem of Pythagoras that:

$$AB^2 + AC^2 = BC^2.$$

So we can write:

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \left(\frac{AC}{BC}\right)^2 + \left(\frac{AB}{BC}\right)^2 \\ &= \frac{AC^2}{BC^2} + \frac{AB^2}{BC^2} \\ &= \frac{AC^2 + AB^2}{BC^2} \\ &= \frac{BC^2}{BC^2} \quad \text{(from Pythagoras)} \\ &= 1\end{aligned}$$

Example 1: *Trigonometric Identities A*

QUESTION

Simplify using identities:

1. $\tan^2\theta \,.\, \cos^2\theta$
2. $\frac{1}{\cos^2\theta} - \tan^2\theta$

SOLUTION

Step 1 : ***Use known identities to replace*** $\tan\theta$

$$\begin{aligned} &= \tan^2\theta \,.\, \cos^2\theta \\ &= \frac{\sin^2\theta}{\cos^2\theta} \,.\, \cos^2\theta \\ &= \sin^2\theta \end{aligned}$$

Step 2 : ***Use known identities to replace*** $\tan\theta$

$$\begin{aligned} &= \frac{1}{\cos^2\theta} - \tan^2\theta \\ &= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{1-\sin^2\theta}{\cos^2\theta} \\ &= \frac{\cos^2\theta}{\cos^2\theta} = 1 \end{aligned}$$

Example 2: *Trigonometric Identities B*

QUESTION

Prove: $\frac{1-\sin x}{\cos x} = \frac{\cos x}{1+\sin x}$

SOLUTION

$$\begin{aligned}
\text{LHS} &= \frac{1-\sin x}{\cos x} \\
&= \frac{1-\sin x}{\cos x} \times \frac{1+\sin x}{1+\sin x} \\
&= \frac{1-\sin^2 x}{\cos x(1+\sin x)} \\
&= \frac{\cos^2 x}{\cos x(1+\sin x)} \\
&= \frac{\cos x}{1+\sin x} = \text{RHS}
\end{aligned}$$

Exercise 17 - 8

1. Simplify the following using the fundamental trigonometric identities:
   (a) $\frac{\cos\theta}{\tan\theta}$
   (b) $\cos^2\theta.\tan^2\theta\ +\ \tan^2\theta.\sin^2\theta$
   (c) $1-\tan^2\theta.\sin^2\theta$
   (d) $1-\sin\theta.\cos\theta.\tan\theta$
   (e) $1-\sin^2\theta$
   (f) $\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)-\cos^2\theta$
2. Prove the following:
   (a) $\frac{1+\sin\theta}{\cos\theta}=\frac{\cos\theta}{1-\sin\theta}$
   (b) $\sin^2\theta+(\cos\theta-\tan\theta)(\cos\theta+\tan\theta)=1-\tan^2\theta$
   (c) $\frac{(2\cos^2\theta-1)}{1}+\frac{1}{(1+\tan^2\theta)}=\frac{2-\tan^2\theta}{1+\tan^2\theta}$
   (d) $\frac{1}{\cos\theta}-\frac{\cos\theta\tan^2\theta}{1}=\cos\theta$
   (e) $\frac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta}=\sin\theta+\cos\theta-\frac{1}{\sin\theta+\cos\theta}$
   (f) $\left(\frac{\cos\theta}{\sin\theta}+\tan\theta\right)\ .\ \cos\theta=\frac{1}{\sin\theta}$

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(1.) 014g (2.) 014h

Reduction Formula

EMBDD

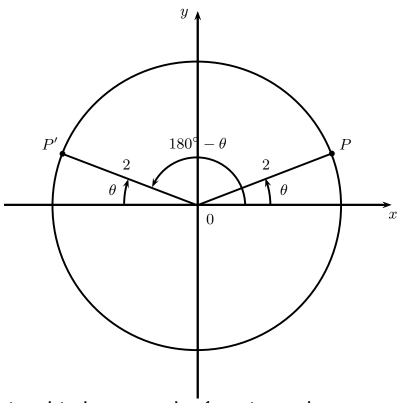
Any trigonometric function whose argument is $90° \pm \theta$; $180° \pm \theta$; $270° \pm \theta$ and $360° \pm \theta$ (hence $-\theta$) can be written simply in terms of θ. For example, you may have noticed that the cosine graph is identical to the sine graph except for a phase shift of $90°$. From this we may expect that $\sin(90° + \theta) = \cos\theta$.

Function Values of $180° \pm \theta$

Activity:	*Reduction Formulae for Function Values of* $180° \pm \theta$

1. **Function Values of** $(180° - \theta)$

   (a) In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has coordinates $(\sqrt{3}; 1)$. If P' is the reflection of P about the y-axis (or the line $x = 0$), use symmetry to write down the coordinates of P'.

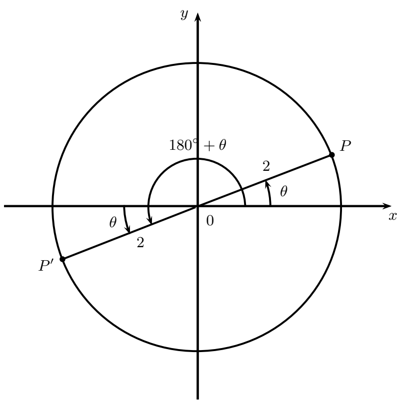
   (b) Write down values for $\sin\theta$, $\cos\theta$ and $\tan\theta$.

   (c) Using the coordinates for P' determine $\sin(180° - \theta)$, $\cos(180° - \theta)$ and $\tan(180° - \theta)$.

   (d) From your results try and determine a relationship between the function values of $(180° - \theta)$ and θ.

2. **Function values of** $(180° + \theta)$

   (a) In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has coordinates $(\sqrt{3}; 1)$. P' is the inversion of P through the origin (reflection about both the x- and y-axes) and lies at an angle of $180° + \theta$ with the x-axis. Write down the coordinates of P'.

   (b) Using the coordinates for P' determine $\sin(180° + \theta)$, $\cos(180° + \theta)$ and $\tan(180° + \theta)$.

   (c) From your results try and determine a relationship between the function values of $(180° + \theta)$ and θ.

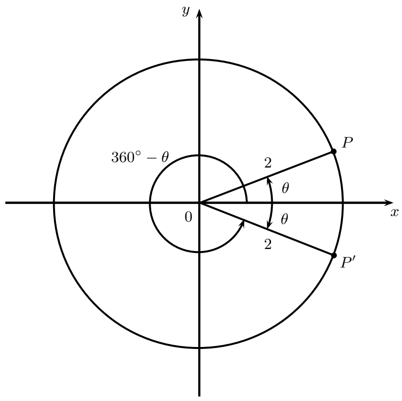
Activity: *Reduction Formulae for Function Values of $360° \pm \theta$*

1. **Function values of** $(360° - \theta)$

   (a) In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has coordinates $(\sqrt{3}; 1)$. P' is the reflection of P about the x-axis or the line $y = 0$. Using symmetry, write down the coordinates of P'.

   (b) Using the coordinates for P′ determine $\sin(360° - \theta)$, $\cos(360° - \theta)$ and $\tan(360° - \theta)$.

   (c) From your results try and determine a relationship between the function values of $(360° - \theta)$ and θ.



It is possible to have an angle which is larger than $360°$. The angle completes one revolution to give $360°$ and then continues to give the required angle. We get the following results:

$$\begin{aligned} \sin(360° + \theta) &= \sin\theta \\ \cos(360° + \theta) &= \cos\theta \\ \tan(360° + \theta) &= \tan\theta \end{aligned}$$

Note also, that if k is any integer, then

$$\begin{aligned} \sin(k360° + \theta) &= \sin\theta \\ \cos(k360° + \theta) &= \cos\theta \\ \tan(k360° + \theta) &= \tan\theta \end{aligned}$$

Example 3: *Basic Use of a Reduction Formula*

QUESTION

Write $\sin 293°$ as the function of an acute angle.

SOLUTION

We note that $293° = 360° - 67°$ thus

$$\begin{aligned} \sin 293° &= \sin(360° - 67°) \\ &= -\sin 67° \end{aligned}$$

where we used the fact that $\sin(360° - \theta) = -\sin\theta$. Check, using your calculator, that these values are in fact equal:

$$\begin{aligned} \sin 293^\circ &= -0{,}92\ldots \\ -\sin 67^\circ &= -0{,}92\ldots \end{aligned}$$

Example 4: *More Complicated*

QUESTION

Evaluate without using a calculator:

$$\tan^2 210^\circ - (1 + \cos 120^\circ)\sin^2 225^\circ$$

SOLUTION

$$\begin{aligned} & \tan^2 210^\circ - (1 + \cos 120^\circ)\sin^2 225^\circ \\ = & \, [\tan(180^\circ + 30^\circ)]^2 - [1 + \cos(180^\circ - 60^\circ)] \,.\, [\sin(180^\circ + 45^\circ)]^2 \\ = & \, (\tan 30^\circ)^2 - [1 + (-\cos 60^\circ)] \,.\, (-\sin 45^\circ)^2 \\ = & \left(\frac{1}{\sqrt{3}}\right)^2 - \left(1 - \frac{1}{2}\right) . \left(-\frac{1}{\sqrt{2}}\right)^2 \\ = & \, \frac{1}{3} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ = & \, \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \end{aligned}$$

Exercise 17 - 9

1. Write these equations as a function of θ only:
   (a) $\sin(180^\circ - \theta)$
   (b) $\cos(180^\circ - \theta)$
   (c) $\cos(360^\circ - \theta)$
   (d) $\cos(360^\circ + \theta)$
   (e) $\tan(180^\circ - \theta)$
   (f) $\cos(360^\circ + \theta)$
2. Write the following trig functions as a function of an acute angle:

(a) $\sin 163°$
(b) $\cos 327°$
(c) $\tan 248°$
(d) $\cos 213°$

3. Determine the following without the use of a calculator:

(a) $(\tan\ 150°)(\sin\ 30°)\ +\ \cos\ 330°$
(b) $(\tan\ 300°)(\cos\ 120°)$
(c) $(1 - \cos\ 30°)(1 - \sin\ 210°)$
(d) $\cos\ 780°\ +\ (\sin\ 315°)(\tan\ 420°)$

4. Determine the following by reducing to an acute angle and using special angles. Do not use a calculator:

(a) $\cos\ 300°$
(b) $\sin\ 135°$
(c) $\cos\ 150°$
(d) $\tan\ 330°$
(e) $\sin\ 120°$
(f) $\tan^2 225°$
(g) $\cos\ 315°$
(h) $\sin^2 420°$
(i) $\tan\ 405°$
(j) $\cos\ 1020°$
(k) $\tan^2 135°$
(l) $1 - \sin^2 210°$

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(1.) 014i (2.) 014j (3.) 014k (4.) 014m

Function Values of $(-\theta)$

When the argument of a trigonometric function is $(-\theta)$ we can add $360°$ without changing the result. Thus for sine and cosine

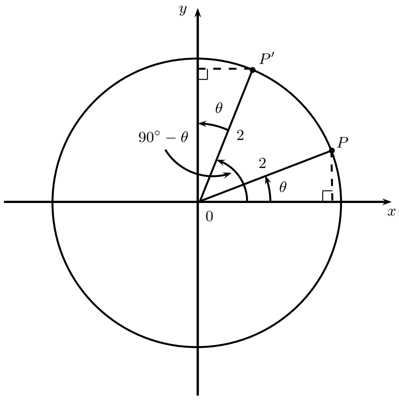
$$\sin(-\theta) = \sin(360° - \theta) = -\sin\theta$$
$$\cos(-\theta) = \cos(360° - \theta) = \cos\theta$$

Function Values of $90° \pm \theta$

Activity:	*Reduction Formulae for Function Values of $90° \pm \theta$*

1. **Function values of $(90° - \theta)$**

(a) In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has coordinates $(\sqrt{3}; 1)$. P' is the reflection of P about the line $y = x$. Using symmetry, write down the coordinates of P'.

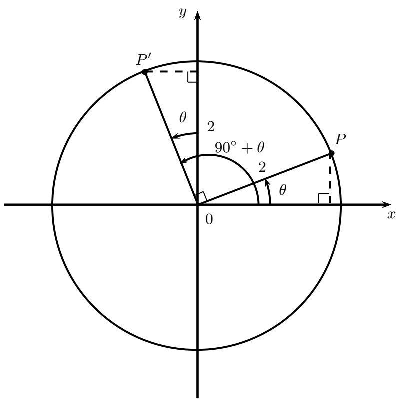


(b) Using the coordinates for P' determine $\sin(90° - \theta)$, $\cos(90° - \theta)$ and $\tan(90° - \theta)$.

(c) From your results try and determine a relationship between the function values of $(90° - \theta)$ and θ.

2. **Function values of** $(90° + \theta)$

(a) In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has coordinates $(\sqrt{3}; 1)$. P' is the rotation of P through $90°$. Using symmetry, write down the coordinates of P'. (Hint: consider P' as the reflection of P about the line $y = x$ followed by a reflection about the y-axis)



(b) Using the coordinates for P' determine $\sin(90° + \theta)$, $\cos(90° + \theta)$ and $\tan(90° + \theta)$.

(c) From your results try and determine a relationship between the function values of $(90° + \theta)$ and θ.

Complementary angles are positive acute angles that add up to $90°$. For example $20°$ and $70°$ are complementary angles.

Sine and cosine are known as *co-functions*. Two functions are called co-functions if $f(A) = g(B)$ whenever $A + B = 90°$ (i.e. A and B are complementary angles). The other trig co-functions are secant and cosecant, and tangent and cotangent.

The function value of an angle is equal to the co-function of its complement (the co-co rule).

Thus for sine and cosine we have

$$\begin{aligned} \sin(90° - \theta) &= \cos\theta \\ \cos(90° - \theta) &= \sin\theta \end{aligned}$$

Example 5: *Co-function Rule*

QUESTION

Write each of the following in terms of $40°$ *using* $\sin(90° - \theta) = \cos\theta$ *and* $\cos(90° - \theta) = \sin\theta$.

1. $\cos 50°$
2. $\sin 320°$
3. $\cos 230°$

SOLUTION

1. $\cos 50° = \sin(90° - 50°) = \sin 40°$
2. $\sin 320° = \sin(360° - 40°) = -\sin 40°$
3. $\cos 230° = \cos(180° + 50°) = -\cos 50°$
   $= -\sin(90° - 50°) = -\sin 40°$

Function Values of $(\theta - 90°)$

$\sin(\theta - 90°) = -\cos\theta$ and $\cos(\theta - 90°) = \sin\theta$.

These results may be proved as follows:

$$\begin{aligned}\sin(\theta - 90°) &= \sin[-(90° - \theta)] \\ &= -\sin(90° - \theta) \\ &= -\cos\theta\end{aligned}$$

similarly, $\cos(\theta - 90°) = \sin\theta$

Summary

The following summary may be made

second quadrant $(180° - \theta)$ **or** $(90° + \theta)$ $\sin(180° - \theta) = +\sin\theta$ $\cos(180° - \theta) = -\cos\theta$ $\tan(180° - \theta) = -\tan\theta$ $\sin(90° + \theta) = +\cos\theta$ $\cos(90° + \theta) = -\sin\theta$	**first quadrant** (θ) **or** $(90° - \theta)$ all trig functions are positive $\sin(360° + \theta) = \sin\theta$ $\cos(360° + \theta) = \cos\theta$ $\tan(360° + \theta) = \tan\theta$ $\sin(90° - \theta) = \sin\theta$ $\cos(90° - \theta) = \cos\theta$
third quadrant $(180° + \theta)$ $\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$ $\tan(180° + \theta) = +\tan\theta$	**fourth quadrant** $(360° - \theta)$ $\sin(360° - \theta) = -\sin\theta$ $\cos(360° - \theta) = +\cos\theta$ $\tan(360° - \theta) = -\tan\theta$

Tip

1. These reduction formulae hold for any angle θ. For convenience, we usually work with θ as if it is acute, i.e. $0° < \theta < 90°$.
2. When determining function values of $180° \pm \theta$, $360° \pm \theta$ and $-\theta$ the functions never change.
3. When determining function values of $90° \pm \theta$ and $\theta - 90°$ the functions changes to its co-function (co-co rule).

Extension: *Function Values of* $(270° \pm \theta)$

Angles in the third and fourth quadrants may be written as $270° \pm \theta$ with θ an acute angle. Similar rules to the above apply. We get

third quadrant $(270° - \theta)$	**fourth quadrant** $(270° + \theta)$
$\sin(270° - \theta) = -\cos\theta$	$\sin(270° + \theta) = -\cos\theta$
$\cos(270° - \theta) = -\sin\theta$	$\cos(270° + \theta) = +\sin\theta$

17.4 Solving Trigonometric Equations

EMBDE

In Grade 10 and 11 we focused on the solution of algebraic equations and excluded equations that dealt with trigonometric functions (i.e. $\sin$ and $\cos$). In this section, the solution of trigonometric equations will be discussed.

The methods described in previous Grades also apply here. In most cases, trigonometric identities will be used to simplify equations, before finding the final solution. The final solution can be found either graphically or using inverse trigonometric functions.

Graphical Solution

EMBDF

As an example, to introduce the methods of solving trigonometric equations, consider

$$\sin\theta = 0{,}5 \quad (17.1)$$

As explained in previous Grades,the solution of Equation 17.1 is obtained by examining the intersecting points of the graphs of:

$$\begin{aligned} y &= \sin\theta \\ y &= 0{,}5 \end{aligned}$$

Both graphs, for $-720° < \theta < 720°$, are shown in Figure 17.9 and the intersection points of the graphs are shown by the dots.

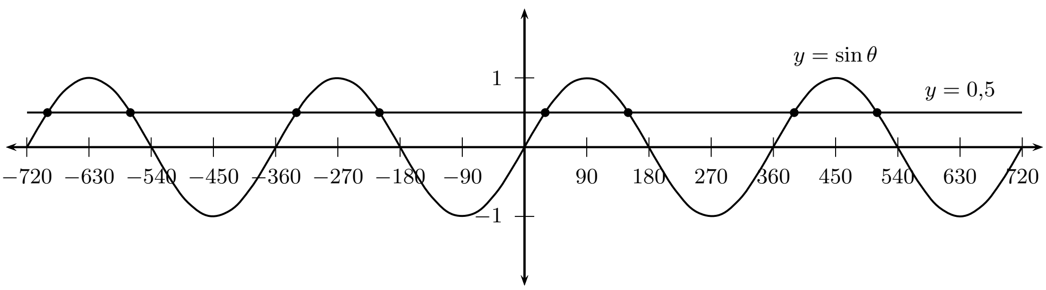


Figure 17.9: Plot of $y = \sin\theta$ and $y = 0{,}5$ showing the points of intersection, hence the solutions to the equation $\sin\theta = 0{,}5$.

In the domain for θ of $-720° < \theta < 720°$, there are eight possible solutions for the equation $\sin\theta = 0{,}5$. These are $\theta = [-690°; -570°; -330°; -210°; 30°; 150°; 390°; 510°]$

Example 6:

QUESTION

Find θ, if $\tan\theta + 0{,}5 = 1{,}5$, with $0° < \theta < 90°$. Determine the solution graphically.

SOLUTION

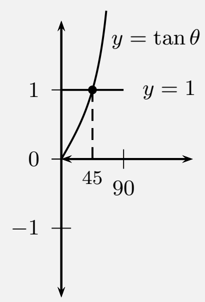
Step 1 : ***Write the equation so that all the terms with the unknown quantity (i.e. θ) are on one side of the equation.***

$$\begin{aligned}\tan\theta + 0{,}5 &= 1{,}5 \\ \tan\theta &= 1\end{aligned}$$

Step 2 : ***Identify the two functions which are intersecting.***

$$\begin{aligned}y &= \tan\theta \\ y &= 1\end{aligned}$$

Step 3 : ***Draw graphs of both functions, over the required domain and identify the intersection point.***



The graphs intersect at $\theta = 45°$.

Algebraic Solution

EMBDG

The inverse trigonometric functions can be used to solve trigonometric equations. These may be shown as second functions on your calculator: sin^{-1}, cos^{-1} and tan^{-1}.

Using inverse trigonometric functions, the equation

$$\sin\theta = 0{,}5$$

is solved as

$$\begin{aligned} \sin\theta &= 0{,}5 \\ &= 30^\circ \end{aligned}$$

On your calculator you would type $\boxed{\sin^{-1}}$ $\boxed{(}$ $\boxed{0{,}5}$ $\boxed{)}$ $\boxed{=}$ to find the size of θ.

This step does not need to be shown in your calculations.

Example 7:

QUESTION

Find θ, if $\tan\theta + 0{,}5 = 1{,}5$, with $0^\circ < \theta < 90^\circ$. Determine the solution using inverse trigonometric functions.

SOLUTION

Step 1 : ***Write the equation so that all the terms with the unknown quantity (i.e. θ) are on one side of the equation. Then solve for the angle using the inverse function.***

$$\begin{aligned} \tan\theta + 0{,}5 &= 1{,}5 \\ \tan\theta &= 1 \\ &= 45^\circ \end{aligned}$$

Trigonometric equations often look very simple. Consider solving the equation $\sin\theta = 0{,}7$. We can take the inverse sine of both sides to find that $\theta = \sin^{-1}(0{,}7)$. If we put this into a calculator we find that $\sin^{-1}(0{,}7) = 44{,}42^\circ$. This is true, however, it does not tell the whole story.

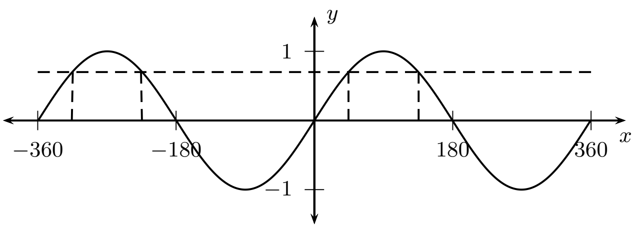


Figure 17.10: The sine graph. The dotted line represents $y = 0{,}7$. There are four points of intersection on this interval, thus four solutions to $\sin\theta = 0{,}7$.

As you can see from Figure 17.10, there are *four* possible angles with a sine of $0{,}7$ between -360° and 360°. If we were to extend the range of the sine graph to infinity we would in fact see that there are an infinite number of solutions to this equation! This difficulty (which is caused by the periodicity of the sine function) makes solving trigonometric equations much harder than they may seem to be.

Any problem on trigonometric equations will require two pieces of information to solve. The first is the equation itself and the second is the *range* in which your answers must lie. The hard part is making sure you find all of the possible answers within the range. Your calculator will always give you the *smallest* answer (*i.e.* the one that lies between $-90°$ and $90°$ for tangent and sine and one between $0°$ and $180°$ for cosine). Bearing this in mind we can already solve trigonometric equations within these ranges.

Example 8:

QUESTION

Find the values of x for which $\sin\left(\frac{x}{2}\right) = 0{,}5$ if it is given that $x < 90°$.

SOLUTION

Because we are told that x is an acute angle, we can simply apply an inverse trigonometric function to both sides.

$$\sin\left(\tfrac{x}{2}\right) = 0{,}5 \qquad (17.2)$$
$$\Rightarrow \tfrac{x}{2} = \arcsin 0{,}5 \qquad (17.3)$$
$$\Rightarrow \tfrac{x}{2} = 30° \qquad (17.4)$$
$$\therefore x = 60° \qquad (17.5)$$

We can, of course, solve trigonometric equations in any range by drawing the graph.

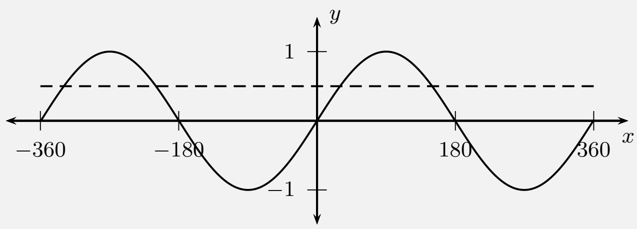
Example 9:

QUESTION

For what values of x does $\sin x = 0{,}5$ when $-360° \leq x \leq 360°$?

SOLUTION

Step 1 : ***Draw the graph***
We take a look at the graph of $\sin x = 0{,}5$ on the interval $[-360°; 360°]$. We want to know when the y value of the graph is $0{,}5$ so we draw in a line at $y = 0{,}5$.

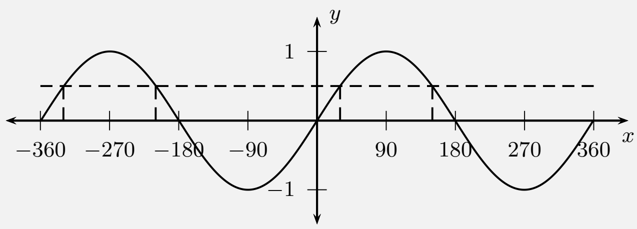


Step 2 :

Notice that this line touches the graph four times. This means that there are four solutions to the equation.

Step 3 :

Read off the x values of those intercepts from the graph as $x = -330°$; $-210°$; $30°$ and $150°$.



This method can be time consuming and inexact. We shall now look at how to solve these problems algebraically.

Solution using CAST diagrams

EMBDH

The Sign of the Trigonometric Function

The first step to finding the trigonometry of any angle is to determine the *sign* of the ratio for a given angle. We shall do this for the sine function first and then do the same for the cosine and tangent.

In Figure 17.11 we have split the sine graph into four *quadrants*, each $90°$ wide. We call them quadrants because they correspond to the four quadrants of the unit circle. We notice from Figure 17.11 that the sine graph is positive in the 1^{st} and 2^{nd} quadrants and negative in the 3^{rd} and 4^{th}. Figure 17.12 shows similar graphs for cosine and tangent.

All of this can be summed up in two ways. Table 17.7 shows which trigonometric functions are positive and which are negative in each quadrant.

A more convenient way of writing this is to note that all functions are positive in the 1^{st} quadrant, only sine is positive in the 2^{nd}, only tangent in the 3^{rd} and only cosine in the 4^{th}. We express this

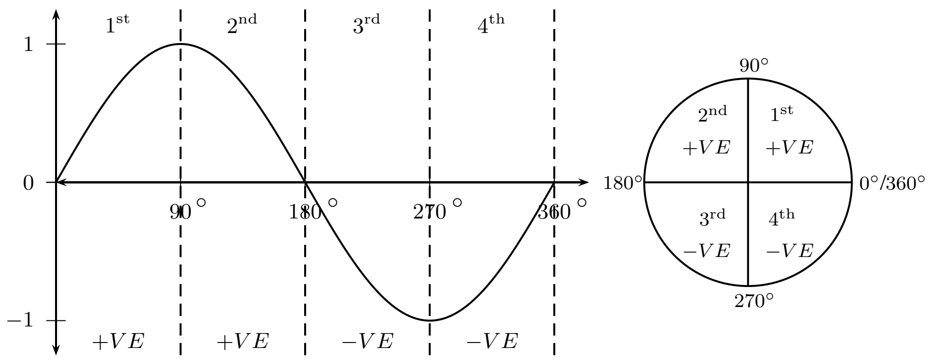


Figure 17.11: The graph and unit circle showing the sign of the sine function.

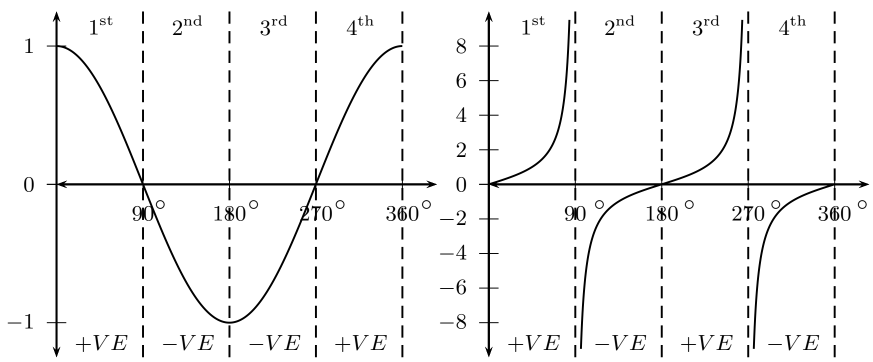


Figure 17.12: Graphs showing the sign of the cosine and tangent functions.

	1^{st}	2^{nd}	3^{rd}	4^{th}
sin	$+VE$	$+VE$	$-VE$	$-VE$
cos	$+VE$	$-VE$	$-VE$	$+VE$
tan	$+VE$	$-VE$	$+VE$	$-VE$

Table 17.7: The signs of the three basic trigonometric functions in each quadrant.

using the CAST diagram (Figure 17.13). This diagram is known as a CAST diagram as the letters, taken anticlockwise from the bottom right, read C-A-S-T. The letter in each quadrant tells us which trigonometric functions are *positive* in that quadrant. The A in the 1^{st} quadrant stands for all (meaning sine, cosine and tangent are all positive in this quadrant). S, C and T, of course, stand for sine, cosine and tangent. The diagram is shown in two forms. The version on the left shows the CAST diagram including the unit circle. This version is useful for equations which lie in large or negative ranges. The simpler version on the right is useful for ranges between $0°$ and $360°$. Another useful diagram shown in Figure 17.13 gives the formulae to use in each quadrant when solving a trigonometric equation.

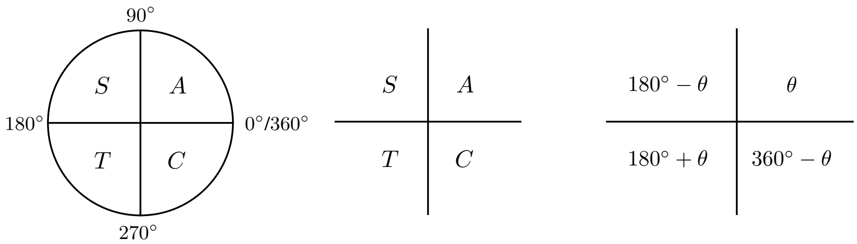


Figure 17.13: The two forms of the CAST diagram and the formulae in each quadrant.

Magnitude of the Trigonometric Functions

Now that we know in which quadrants our solutions lie, we need to know which angles in these quadrants satisfy our equation.
Calculators give us the smallest possible answer (sometimes negative) which satisfies the equation. For example, if we wish to solve $\sin\theta = 0{,}3$ we can apply the inverse sine function to both sides of the equation to find:

$$\begin{aligned}\sin\theta &= 0{,}3\\ \therefore \theta &= 17{,}46°\end{aligned}$$

However, we know that this is just one of infinitely many possible answers. We get the rest of the answers by finding relationships between this small angle, θ, and answers in other quadrants.
To do this we use our small angle θ as a *reference angle*. We then look at the sign of the trigonometric function in order to decide in which quadrants we need to work (using the CAST diagram) and add multiples of the period to each, remembering that sine, cosine and tangent are periodic (repeating) functions. To add multiples of the period we use $(360° \,.\, n)$ (where n is an integer) for sine and cosine and $(180° \,.\, n)$; $n \in \mathbb{Z}$, for the tangent.

Example 10:

QUESTION

Solve for θ:

$$\sin\theta = 0{,}3$$

SOLUTION

Step 1 : ***Determine in which quadrants the solution lies***
We look at the sign of the trigonometric function. $\sin\theta$ is given as a positive amount (0,3). Reference to the CAST diagram shows that sine is positive in the first and second quadrants.

S	A
T	C

Step 2 : ***Determine the reference angle***
The small angle θ is the angle returned by the calculator:

$$\begin{aligned} \sin\theta &= 0{,}3 \\ \therefore \theta &= 17{,}46^\circ \end{aligned}$$

Step 3 : ***Determine the general solution***
Our solution lies in quadrants I and II. We therefore use θ and $180^\circ - \theta$, and add the $(360^\circ \,.\, n)$ for the periodicity of sine.

$180^\circ - \theta$	θ
$180^\circ + \theta$	$360^\circ - \theta$

$$\begin{aligned} \text{I}: \quad \theta &= 17{,}46^\circ + (360^\circ \,.\, n);\ n \in \mathbb{Z} \\ \text{II}: \quad \theta &= 180^\circ - 17{,}46^\circ + (360^\circ \,.\, n);\ n \in \mathbb{Z} \\ &= 162{,}54^\circ + (360^\circ \,.\, n);\ n \in \mathbb{Z} \end{aligned}$$

This is called the *general solution*.

Step 4 : ***Find the specific solutions***
We can then find *all* the values of θ by substituting $n = \ldots - 1;\ 0;\ 1;\ 2; \ldots$etc.
For example,
If $n = 0, \quad \theta = 17{,}46^\circ; 162{,}54^\circ$
If $n = 1, \quad \theta = 377{,}46^\circ; 522{,}54^\circ$
If $n = -1, \quad \theta = -342{,}54^\circ; -197{,}46^\circ$
We can find as many as we like or find specific solutions in a given interval by choosing more values for n.

General Solution Using Periodicity

EMBDI

Up until now we have only solved trigonometric equations where the argument (the bit after the function, e.g. the θ in $\cos\theta$ or the $(2x-7)$ in $\tan(2x-7)$), has been θ. If there is anything more complicated than this we need to be a little more careful.

Let us try to solve $\tan(2x - 10^\circ) = 2{,}5$ in the range $-360^\circ \leq x \leq 360^\circ$. We want solutions for positive tangent so using our CAST diagram we know to look in the 1^{st} and 3^{rd} quadrants. Our calculator tells us that $2x - 10^\circ = 68{,}2^\circ$. This is our reference angle. So to find the general solution we proceed as follows:

$$\begin{aligned} \tan(2x - 10^\circ) &= 2{,}5 \\ \therefore 2x - 10^\circ &= 68{,}2^\circ \\ \text{I}: \quad 2x - 10^\circ &= 68{,}2^\circ + (180^\circ \,.\, n) \\ 2x &= 78{,}2^\circ + (180^\circ \,.\, n) \\ x &= 39{,}1^\circ + (90^\circ \,.\, n);\ n \in \mathbb{Z} \end{aligned}$$

This is the general solution. Notice that we added the $10°$ and divided by 2 only at the end. Notice that we added $(180° \,.\, n)$ because the tangent has a period of $180°$. This is **also** divided by 2 in the last step to keep the equation balanced. We chose quadrants I and III because $\tan$ was positive and we used the formulae θ in quadrant I and $(180° + \theta)$ in quadrant III. To find solutions where $-360° < x < 360°$ we substitute integers for n:

- $n = 0$; $x = 39{,}1°$; $219{,}1°$
- $n = 1$; $x = 129{,}1°$; $309{,}1°$
- $n = 2$; $x = 219{,}1°$; $399{,}1°$ (too big!)
- $n = 3$; $x = 309{,}1°$; $489{,}1°$ (too big!)
- $n = -1$; $x = -50{,}9°$; $129{,}1°$
- $n = -2$; $x = -140{,}9°$; $-39{,}9°$
- $n = -3$; $x = -230{,}9°$; $-50{,}9°$
- $n = -4$; $x = -320{,}9°$; $-140{,}9°$
- $n = -5$; $x = -410{,}9°$; $-230{,}9°$
- $n = -6$; $x = -500{,}9°$; $-320{,}9°$

Solution: $x = -320{,}9°; -230°; -140{,}9°; -50{,}9°; 39{,}1°; 129{,}1°; 219{,}1°$ and $309{,}1°$

Linear Trigonometric Equations

EMBDJ

Just like with regular equations without trigonometric functions, solving trigonometric equations can become a lot more complicated. You should solve these just like normal equations to isolate a single trigonometric ratio. Then you follow the strategy outlined in the previous section.

Example 11:

QUESTION

Write down the general solution for $3\cos(\theta - 15°) - 1 = -2{,}583$

SOLUTION

$$\begin{aligned}
3\cos(\theta - 15^\circ) - 1 &= -2{,}583 \\
3\cos(\theta - 15^\circ) &= -1{,}583 \\
\cos(\theta - 15^\circ) &= -0{,}5276\ldots \\
\text{reference angle: } (\theta - 15^\circ) &= 58{,}2^\circ \\
\text{II}: \quad \theta - 15^\circ &= 180^\circ - 58{,}2^\circ + (360^\circ \,.\, n);\ n \in \mathbb{Z} \\
\theta &= 136{,}8^\circ + (360^\circ \,.\, n);\ n \in \mathbb{Z} \\
\text{III}: \quad \theta - 15^\circ &= 180^\circ + 58{,}2^\circ + (360^\circ \,.\, n);\ n \in \mathbb{Z} \\
\theta &= 253{,}2^\circ + (360^\circ \,.\, n);\ n \in \mathbb{Z}
\end{aligned}$$

Quadratic and Higher Order Trigonometric Equations EMBDK

The simplest quadratic trigonometric equation is of the form

$$\sin^2 x - 2 = -1{,}5$$

This type of equation can be easily solved by rearranging to get a more familiar linear equation

$$\begin{aligned}
\sin^2 x &= 0{,}5 \\
\Rightarrow \sin x &= \pm\sqrt{0{,}5}
\end{aligned}$$

This gives two linear trigonometric equations. The solutions to either of these equations will satisfy the original quadratic.

The next level of complexity comes when we need to solve a trinomial which contains trigonometric functions. It is much easier in this case to use *temporary variables*. Consider solving

$$\tan^2(2x + 1) + 3\tan(2x + 1) + 2 = 0$$

Here we notice that $\tan(2x + 1)$ occurs twice in the equation, hence we let $y = \tan(2x + 1)$ and rewrite:

$$y^2 + 3y + 2 = 0$$

That should look rather more familiar. We can immediately write down the factorised form and the solutions:

$$\begin{aligned}
(y + 1)(y + 2) &= 0 \\
\Rightarrow y = -1 \quad \text{OR} \quad y &= -2
\end{aligned}$$

Next we just substitute back for the temporary variable:

$$\tan(2x + 1) = -1 \quad \text{or} \quad \tan(2x + 1) = -2$$

And then we are left with two linear trigonometric equations. Be careful: sometimes one of the two solutions will be outside the *range* of the trigonometric function. In that case you need to discard that solution. For example consider the same equation with cosines instead of tangents

$$\cos^2(2x+1)+3\cos(2x+1)+2=0$$

Using the same method we find that

$$\cos(2x+1)=-1 \quad \text{or} \quad \cos(2x+1)=-2$$

The second solution cannot be valid as cosine must lie between -1 and 1. We must, therefore, reject the second equation. Only solutions to the first equation will be valid.

More Complex Trigonometric Equations

EMBDL

Here are two examples on the level of the hardest trigonometric equations you are likely to encounter. They require using everything that you have learnt in this chapter. If you can solve these, you should be able to solve anything!

Example 12:

QUESTION

Solve $2\cos^2 x - \cos x - 1 = 0$ *for* $x \in [-180°; 360°]$

SOLUTION

Step 1 : ***Use a temporary variable***

We note that $\cos x$ occurs twice in the equation. So, let $y = \cos x$. Then we have $2y^2 - y - 1 = 0$ Note that with practise you may be able to leave out this step.

Step 2 : ***Solve the quadratic equation***

Factorising yields

$$(2y+1)(y-1)=0$$
$$\therefore \quad y=-0{,}5 \quad \text{or} \quad y=1$$

Step 3 : ***Substitute back and solve the two resulting equations***

We thus get

$$\cos x=-0{,}5 \quad \text{or} \quad \cos x=1$$

Both equations are valid (*i.e.* lie in the range of cosine).
General solution:

$$\begin{aligned} \cos x &= -0{,}5 \quad [60^\circ] \\ \text{II}: \quad x &= 180^\circ - 60^\circ + (360^\circ . n); n \in \mathbb{Z} \\ &= 120^\circ + (360^\circ . n); n \in \mathbb{Z} \\ \text{III}: \quad x &= 180^\circ + 60^\circ (360^\circ . n); n \in \mathbb{Z} \\ &= 240^\circ + (360^\circ . n); n \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \cos x &= 1 \quad [90^\circ] \\ \text{I; IV}: \quad x &= 0^\circ (360^\circ . n); n \in \mathbb{Z} \\ &= (360^\circ . n); n \in \mathbb{Z} \end{aligned}$$

Now we find the specific solutions in the interval $[-180^\circ; 360^\circ]$. Appropriate values of n yield

$$x = -120^\circ; 0^\circ; 120^\circ; 240^\circ; 360^\circ$$

Example 13:

QUESTION

Solve for x in the interval $[-360^\circ; 360^\circ]$:

$$2\sin^2 x - \sin x \cos x = 0$$

SOLUTION

Step 1 : ***Factorise***

Factorising yields

$$\sin x(2\sin x - \cos x) = 0$$

which gives two equations

$$\sin x = 0$$

$$\begin{aligned} 2\sin - \cos x &= 0 \\ 2\sin x &= \cos x \\ \frac{2\sin x}{\cos x} &= \frac{\cos x}{\cos x} \\ 2\tan x &= 1 \\ \tan x &= \tfrac{1}{2} \end{aligned}$$

Step 2 : ***Solve the two trigonometric equations***

General solution:

$$\begin{aligned} \sin x &= 0 \quad [0^\circ] \\ \therefore \quad x &= (180^\circ . n); n \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \tan x &= \tfrac{1}{2} \quad [26{,}57^\circ] \\ \text{I; III}: \quad x &= 26{,}57^\circ + (180^\circ . n); n \in \mathbb{Z} \end{aligned}$$

Specific solution in the interval $[-360^\circ; 360^\circ]$:

$$x = -360^\circ; -206{,}57^\circ; -180^\circ; -26{,}57^\circ; 0^\circ; 26{,}57^\circ; 180^\circ; 206{,}25^\circ; 360^\circ$$

Exercise 17 - 10

1. (a) Find the general solution of each of the following equations. Give answers to one decimal place.
   (b) Find all solutions in the interval $\theta \in [-180°; 360°]$.
      i. $\sin\theta = -0{,}327$
      ii. $\cos\theta = 0{,}231$
      iii. $\tan\theta = -1{,}375$
      iv. $\sin\theta = 2{,}439$
2. (a) Find the general solution of each of the following equations. Give answers to one decimal place.
   (b) Find all solutions in the interval $\theta \in [0°; 360°]$.
      i. $\cos\theta = 0$
      ii. $\sin\theta = \frac{\sqrt{3}}{2}$
      iii. $2\cos\theta - \sqrt{3} = 0$
      iv. $\tan\theta = -1$
      v. $5\cos\theta = -2$
      vi. $3\sin\theta = -1{,}5$
      vii. $2\cos\theta + 1{,}3 = 0$
      viii. $0{,}5\tan\theta + 2{,}5 = 1{,}7$
3. (a) Write down the general solution for x if $\tan x = -1{,}12$.
   (b) Hence determine values of $x \in [-180°; 180°]$.
4. (a) Write down the general solution for θ if $\sin\theta = -0{,}61$.
   (b) Hence determine values of $\theta \in [0°; 720°]$.
5. (a) Solve for A if $\sin(A + 20°) = 0{,}53$
   (b) Write down the values of $A \in [0°; 360°]$
6. (a) Solve for x if $\cos(x + 30°) = 0{,}32$
   (b) Write down the values of $x \in [-180°; 360°]$
7. (a) Solve for θ if $\sin^2(\theta) + 0{,}5\sin\theta = 0$
   (b) Write down the values of $\theta \in [0°; 360°]$

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(1.) 014n (2.) 014p (3.) 014q (4.) 014r (5.) 014s (6.) 014t
(7.) 014u

17.5 Sine and Cosine Identities

EMBDM

There are a few identities relating to the trigonometric functions that make working with triangles easier. These are:

1. the sine rule
2. the cosine rule
3. the area rule

and will be described and applied in this section.

The Sine Rule

EMBDN

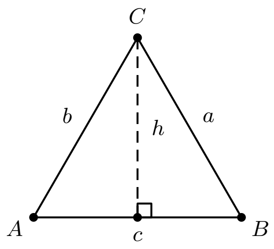
DEFINITION: The Sine Rule

The sine rule applies to any triangle:

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$

where a is the side opposite $\hat{A}$, b is the side opposite $\hat{B}$ and c is the side opposite $\hat{C}$.

Consider $\triangle ABC$.



The area of $\triangle ABC$ can be written as:

$$\text{area} \quad \triangle ABC = \frac{1}{2} c \,.\, h.$$

However, h can be calculated in terms of $\hat{A}$ or $\hat{B}$ as:

$$\begin{aligned} \sin \hat{A} &= \frac{h}{b} \\ \therefore \quad h &= b \,.\, \sin \hat{A} \end{aligned}$$

and

$$\begin{aligned} \sin \hat{B} &= \frac{h}{a} \\ \therefore \quad h &= a \,.\, \sin \hat{B} \end{aligned}$$

Therefore the area of $\triangle ABC$ is:

$$\begin{aligned} & \frac{1}{2}c\,.\,h \\ = & \frac{1}{2}c\,.\,b\,.\,\sin\hat{A} \\ = & \frac{1}{2}c\,.\,a\,.\,\sin\hat{B} \end{aligned}$$

Similarly, by drawing the perpendicular between point B and line AC we can show that:

$$\frac{1}{2}c\,.\,b\,.\,\sin\hat{A} = \frac{1}{2}a\,.\,b\,.\,\sin\hat{C}$$

Therefore the area of $\triangle ABC$ is:

$$\frac{1}{2}c\,.\,b\,.\,\sin\hat{A} = \frac{1}{2}c\,.\,a\,.\,\sin\hat{B} = \frac{1}{2}a\,.\,b\,.\,\sin\hat{C}$$

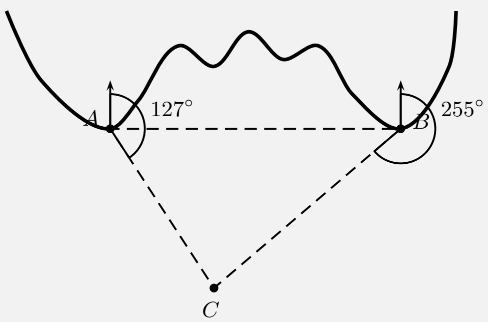
If we divide through by $\frac{1}{2}a\,.\,b\,.\,c$, we get:

$$\frac{\sin\hat{A}}{a} = \frac{\sin\hat{B}}{b} = \frac{\sin\hat{C}}{c}$$

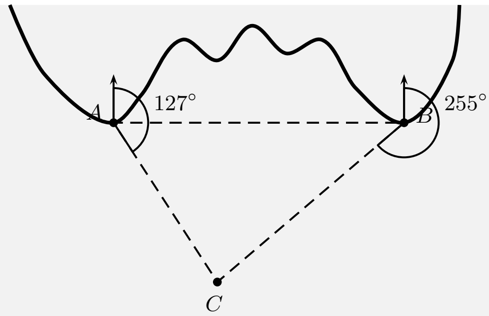
This is known as the sine rule and applies to *any* triangle, right-angled or not.

Example 14: *Lighthouses*

QUESTION

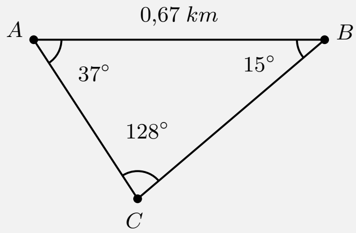


There is a coastline with two lighthouses, one on either side of a beach. The two lighthouses are $0{,}67$ *km apart and one is exactly due east of the other. The lighthouses tell how close a boat is by taking bearings to the boat (remember – a bearing is an angle measured clockwise from north). These bearings are shown. Use the sine rule to calculate how far the boat is from each lighthouse.*



SOLUTION

We can see that the two lighthouses and the boat form a triangle. Since we know the distance between the lighthouses and we have two angles we can use trigonometry to find the remaining two sides of the triangle, the distance of the boat from the two lighthouses.



We need to know the lengths of the two sides AC and BC. We can use the sine rule to find our missing lengths.

$$\begin{aligned}\frac{BC}{\sin\hat{A}} &= \frac{AB}{\sin\hat{C}}\\ BC &= \frac{AB\,.\,\sin\hat{A}}{\sin\hat{C}}\\ &= \frac{(0{,}67)\sin(37^\circ)}{\sin(128^\circ)}\\ &= 0{,}51\text{ km}\end{aligned}$$

$$\begin{aligned}\frac{AC}{\sin\hat{B}} &= \frac{AB}{\sin\hat{C}}\\ AC &= \frac{AB\,.\,\sin\hat{B}}{\sin\hat{C}}\\ &= \frac{(0{,}67)\sin(15^\circ)}{\sin(128^\circ)}\\ &= 0{,}22\text{ km}\end{aligned}$$

Exercise 17 - 11

1. Show that
$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$
is equivalent to:
$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$
Note: either of these two forms can be used.

2. Find all the unknown sides and angles of the following triangles:
   (a) $\triangle PQR$ in which $\hat{Q} = 64°$; $\hat{R} = 24°$ and $r = 3$
   (b) $\triangle KLM$ in which $\hat{K} = 43°$; $\hat{M} = 50°$ and $m = 1$
   (c) $\triangle ABC$ in which $\hat{A} = 32{,}7°$; $\hat{C} = 70{,}5°$ and $a = 52{,}3$
   (d) $\triangle XYZ$ in which $\hat{X} = 56°$; $\hat{Z} = 40°$ and $x = 50$

3. In $\triangle ABC$, $\hat{A} = 116°$; $\hat{C} = 32°$ and $AC = 23\ \text{m}$. Find the length of the side AB.

4. In $\triangle RST$, $\hat{R} = 19°$; $\hat{S} = 30°$ and $RT = 120$ km. Find the length of the side ST.

5. In $\triangle KMS$, $\hat{K} = 20°$; $\hat{M} = 100°$ and $s = 23$ cm. Find the length of the side m.

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(1.) 014v (2.) 014w (3.) 014x (4.) 014y (5.) 014z

The Cosine Rule

EMBDO

DEFINITION: The Cosine Rule

The cosine rule applies to any triangle and states that:

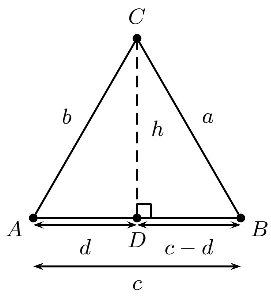
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos \hat{A} \\ b^2 &= c^2 + a^2 - 2ca \cos \hat{B} \\ c^2 &= a^2 + b^2 - 2ab \cos \hat{C} \end{aligned}$$

where a is the side opposite $\hat{A}$, b is the side opposite $\hat{B}$ and c is the side opposite $\hat{C}$.

The cosine rule relates the length of a side of a triangle to the angle opposite it and the lengths of the other two sides.

Consider $\triangle ABC$ which we will use to show that:

$$a^2 = b^2 + c^2 - 2bc \cos \hat{A}.$$



In $\triangle DCB$:

$$a^2 = (c-d)^2 + h^2 \tag{17.6}$$

from the theorem of Pythagoras.

In $\triangle ACD$:

$$b^2 = d^2 + h^2 \tag{17.7}$$

from the theorem of Pythagoras.

We can eliminate h^2 from (17.6) and (17.7) to get:

$$\begin{aligned} b^2 - d^2 &= a^2 - (c-d)^2 \\ a^2 &= b^2 + (c^2 - 2cd + d^2) - d^2 \\ &= b^2 + c^2 - 2cd + d^2 - d^2 \\ &= b^2 + c^2 - 2cd \qquad (17.8) \end{aligned}$$

In order to eliminate d we look at $\triangle ACD$, where we have:

$$\cos \hat{A} = \frac{d}{b}.$$

So,

$$d = b \cos \hat{A}.$$

Substituting this into (17.8), we get:

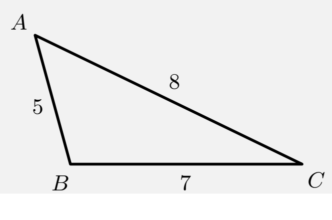
$$a^2 = b^2 + c^2 - 2bc \cos \hat{A} \tag{17.9}$$

The other cases can be proved in an identical manner.

Example 15:

QUESTION

Find $\hat{A}$:



SOLUTION

Applying the cosine rule:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\hat{A} \\ \therefore \quad \cos\hat{A} &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{8^2 + 5^2 - 7^2}{2 \,.\, 8 \,.\, 5} \\ &= 0{,}5 \\ \therefore \quad \hat{A} &= 60^\circ \end{aligned}$$

Exercise 17 - 12

1. Solve the following triangles *i.e.* find all unknown sides and angles
   (a) $\triangle ABC$ in which $\hat{A} = 70^\circ$; $b = 4$ and $c = 9$
   (b) $\triangle XYZ$ in which $\hat{Y} = 112^\circ$; $x = 2$ and $y = 3$
   (c) $\triangle RST$ in which $RS = 2$; $ST = 3$ and $RT = 5$
   (d) $\triangle KLM$ in which $KL = 5$; $LM = 10$ and $KM = 7$
   (e) $\triangle JHK$ in which $\hat{H} = 130^\circ$; $JH = 13$ and $HK = 8$
   (f) $\triangle DEF$ in which $d = 4$; $e = 5$ and $f = 7$
2. Find the length of the third side of the $\triangle XYZ$ where:
   (a) $\hat{X} = 71{,}4^\circ$; $y = 3{,}42$ km and $z = 4{,}03$ km
   (b) ; $x = 103{,}2$ cm; $\hat{Y} = 20{,}8^\circ$ and $z = 44{,}59$ cm
3. Determine the largest angle in:
   (a) $\triangle JHK$ in which $JH = 6$; $HK = 4$ and $JK = 3$
   (b) $\triangle PQR$ where $p = 50$; $q = 70$ and $r = 60$

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(1.) 0150 (2.) 0151 (3.) 0152

The Area Rule

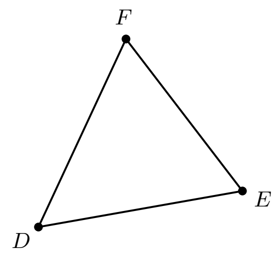
EMBDP

DEFINITION: The Area Rule

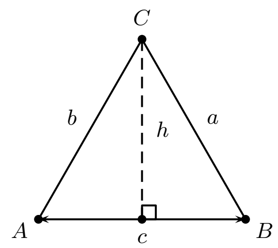
The area rule applies to any triangle and states that the area of a triangle is given by half the product of any two sides with the sine of the angle between them.

That means that in the $\triangle DEF$, the area is given by:

$$\begin{aligned} A &= \frac{1}{2}DE\,.\,EF\sin\hat{E} \\ &= \frac{1}{2}EF\,.\,FD\sin\hat{F} \\ &= \frac{1}{2}FD\,.\,DE\sin\hat{D} \end{aligned}$$



In order show that this is true for all triangles, consider $\triangle ABC$.



The area of any triangle is half the product of the base and the perpendicular height. For $\triangle ABC$, this is:

$$A = \frac{1}{2}c\,.\,h.$$

However, h can be written in terms of $\hat{A}$ as:

$$h = b\sin\hat{A}$$

So, the area of $\triangle ABC$ is:

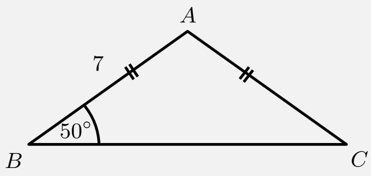
$$A = \frac{1}{2}c\,.\,b\sin\hat{A}.$$

Using an identical method, the area rule can be shown for the other two angles.

Example 16: The Area Rule

QUESTION

Find the area of $\triangle ABC$:



SOLUTION

$\triangle ABC$ is isosceles, therefore $AB = AC = 7$ and $\hat{C} = \hat{B} = 50°$. Hence $\hat{A} = 180° - 50° - 50° = 80°$. Now we can use the area rule to find the area:

$$\begin{aligned} A &= \frac{1}{2} cb \sin \hat{A} \\ &= \frac{1}{2} \,.\, 7 \,.\, 7 \,.\, \sin 80° \\ &= 24{,}13 \end{aligned}$$

Exercise 17 - 13

Draw sketches of the figures you use in this exercise.

1. Find the area of $\triangle PQR$ in which:
   (a) $\hat{P} = 40°$; $q = 9$ and $r = 25$
   (b) $\hat{Q} = 30°$; $r = 10$ and $p = 7$
   (c) $\hat{R} = 110°$; $p = 8$ and $q = 9$
2. Find the area of:
   (a) $\triangle XYZ$ with $XY = 6$ cm; $XZ = 7$ cm and $\hat{Z} = 28°$
   (b) $\triangle PQR$ with $PR = 52$ cm; $PQ = 29$ cm and $\hat{P} = 58{,}9°$
   (c) $\triangle EFG$ with $FG = 2{,}5$ cm; $EG = 7{,}9$ cm and $\hat{G} = 125°$
3. Determine the area of a parallelogram in which two adjacent sides are 10 cm and 13 cm and the angle between them is $55°$.
4. If the area of $\triangle ABC$ is $5000\,\text{m}^2$ with $a = 150\,\text{m}$ and $b = 70\,\text{m}$, what are the two possible sizes of $\hat{C}$?

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(1.) 0153 (2.) 0154 (3.) 0155 (4.) 0156

Summary of the Trigonometric Rules and Identities

EMBDQ

Squares Identity	Quotient Identity
$\cos^2\theta + \sin^2\theta = 1$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$

Odd/Even Identities	Periodicity Identities	Cofunction Identities
$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$	$\sin(\theta \pm 360°) = \sin\theta$ $\cos(\theta \pm 360°) = \cos\theta$	$\sin(90° - \theta) = \cos\theta$ $\cos(90° - \theta) = \sin\theta$

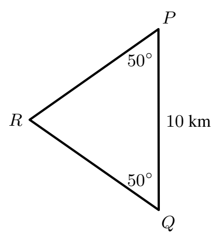
Sine Rule	Area Rule	Cosine Rule
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	Area $= \frac{1}{2}bc\cos A$ Area $= \frac{1}{2}ac\cos B$ Area $= \frac{1}{2}ab\cos C$	$a^2 = b^2 + c^2 - 2bc\cos A$ $b^2 = a^2 + c^2 - 2ac\cos B$ $c^2 = a^2 + b^2 - 2ab\cos C$

Chapter 17 End of Chapter Exercises

Q is a ship at a point 10 km due South of another ship P. R is a lighthouse on the coast such that $\hat{P} = \hat{Q} = 50°$.
Determine:

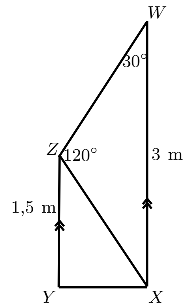
(a) the distance QR

(b) the shortest distance from the lighthouse to the line joining the two ships (PQ).



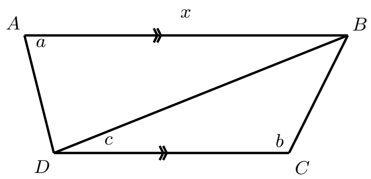
1. $WXYZ$ is a trapezium ($WX \parallel XZ$) with $WX = 3\ \text{m}$; $YZ = 1{,}5\ \text{m}$; $\hat{Z} = 120°$ and $\hat{W} = 30°$.

   Determine the distances XZ and XY.

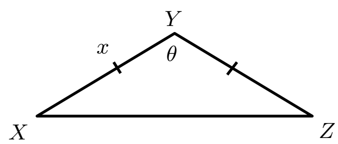


2. On a flight from Johannesburg to Cape Town, the pilot discovers that he has been flying $3°$ off course. At this point the plane is 500 km from Johannesburg. The direct distance between Cape Town and Johannesburg airports is $1\ 552$ km. Determine, to the nearest km:
   (a) The distance the plane has to travel to get to Cape Town and hence the extra distance that the plane has had to travel due to the pilot's error.
   (b) The correction, to one hundredth of a degree, to the plane's heading (or direction).

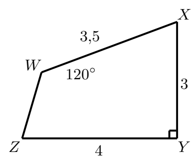
3. $ABCD$ is a trapezium (*i.e.* $AB \parallel CD$). $AB = x$; $B\hat{A}D = a$; $B\hat{C}D = b$ and $B\hat{D}C = c$.
   Find an expression for the length of CD in terms of x, a, b and c.



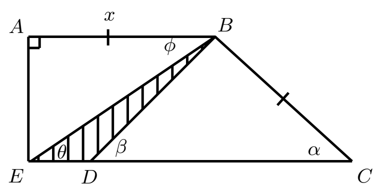
4. A surveyor is trying to determine the distance between points X and Z. However the distance cannot be determined directly as a ridge lies between the two points. From a point Y which is equidistant from X and Z, he measures the angle $X\hat{Y}Z$.
   (a) If $XY = x$ and $X\hat{Y}Z = \theta$, show that $XZ = x\sqrt{2(1 - \cos\theta)}$.
   (b) Calculate XZ (to the nearest kilometre) if $x = 240$ km and $\theta = 132°$.



5. Find the area of $WXYZ$ (to two decimal places):



6. Find the area of the shaded triangle in terms of x, α, β, θ and ϕ:



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(1.) 0157 (2.) 0158 (3.) 0159 (4.) 015a (5.) 015b (6.) 015c
(7.) 015d

Statistics 18

18.1 Introduction

EMBDR

This chapter gives you an opportunity to build on what you have learned in previous grades about data handling and probability. The work done will be mostly of a practical nature. Through problem solving and activities, you will end up mastering further methods of collecting, organising, displaying and analysing data. You will also learn how to interpret data, and not always to accept the data at face value, because data is sometimes misused and abused in order to try to falsely prove or support a viewpoint. Measures of central tendency (mean, median and mode) and dispersion (range, percentiles, quartiles, inter-quartile, semi-inter-quartile range, variance and standard deviation) will be investigated. Of course, the activities involving probability will be familiar to most of you - for example, you may have played dice or card games even before you came to school. Your basic understanding of probability and chance gained so far will deepen to enable you to come to a better understanding of how chance and uncertainty can be measured and understood.

See introductory video: VMfvd at www.everythingmaths.co.za

18.2 Standard Deviation and Variance

EMBDS

The measures of central tendency (mean, median and mode) and measures of dispersion (quartiles, percentiles, ranges) provide information on the data values at the centre of the data set and provide information on the spread of the data. The information on the spread of the data is however based on data values at specific points in the data set, e.g. the end points for range and data points that divide the data set into four equal groups for the quartiles. The behaviour of the entire data set is therefore not examined.

A method of determining the spread of data is by calculating a measure of the possible distances between the data and the mean. The two important measures that are used are called the *variance* and the *standard deviation* of the data set.

Variance

EMBDT

The variance of a data set is the average squared distance between the mean of the data set and each data value. An example of what this means is shown in Figure 18.1. The graph represents the results of 100 tosses of a fair coin, which resulted in 45 heads and 55 tails. The mean of the results is 50. The squared distance between the heads value and the mean is $(45-50)^2=25$ and the squared distance between the tails value and the mean is $(55-50)^2=25$. The average of these two squared distances gives the variance, which is $\frac{1}{2}(25+25)=25$.

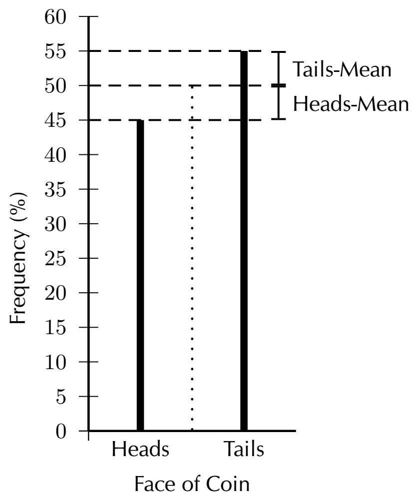

Figure 18.1: The graph shows the results of 100 tosses of a fair coin, with 45 heads and 55 tails. The mean value of the tosses is shown as a vertical dotted line. The difference between the mean value and each data value is shown.

Population Variance

Let the population consist of n elements $\{x_1; x_2; \ldots; x_n\}$, with mean $\bar{x}$ (read as "x bar"). The variance of the population, denoted by σ^2, is the average of the square of the distance of each data value from the mean value.

$$\sigma^2 = \frac{(\sum(x - \bar{x}))^2}{n}. \quad (18.1)$$

Since the population variance is squared, it is not directly comparable with the mean and the data themselves.

Sample Variance

Let the sample consist of the n elements $\{x_1, x_2, \ldots, x_n\}$, taken from the population, with mean $\bar{x}$. The variance of the sample, denoted by s^2, is the average of the squared deviations from the sample mean:

$$s^2 = \frac{\sum(x - \bar{x})^2}{n-1}. \quad (18.2)$$

Since the sample variance is squared, it is also not directly comparable with the mean and the data themselves.

A common question at this point is "Why is the numerator squared?" One answer is: to get rid of the negative signs. Numbers are going to fall above and below the mean and, since the variance is looking for distance, it would be counterproductive if those distances factored each other out.

Difference between Population Variance and Sample Variance

As seen a distinction is made between the variance, σ^2, of a whole population and the variance, s^2 of a sample extracted from the population.

When dealing with the complete population the (population) variance is a constant, a parameter which helps to describe the population. When dealing with a sample from the population the (sample) variance varies from sample to sample. Its value is only of interest as an estimate for the population variance.

Properties of Variance

The variance is never negative because the squares are always positive or zero. The unit of variance is the square of the unit of observation. For example, the variance of a set of heights measured in centimetres will be given in square centimeters. This fact is inconvenient and has motivated many statisticians to instead use the square root of the variance, known as the standard deviation, as a summary of dispersion.

Standard Deviation

EMBDU

Since the variance is a squared quantity, it cannot be directly compared to the data values or the mean value of a data set. It is therefore more useful to have a quantity which is the square root of the variance. This quantity is known as the standard deviation.

In statistics, the standard deviation is the most common measure of statistical dispersion. Standard deviation measures how spread out the values in a data set are. More precisely, it is a measure of the average distance between the values of the data in the set and the mean. If the data values are all similar, then the standard deviation will be low (closer to zero). If the data values are highly variable, then the standard variation is high (further from zero).

The standard deviation is always a positive number and is always measured in the same units as the original data. For example, if the data are distance measurements in metres, the standard deviation will also be measured in metres.

Population Standard Deviation

Let the population consist of n elements $\{x_1; x_2; \ldots; x_n\}$, with mean $\bar{x}$. The standard deviation of the population, denoted by σ, is the square root of the average of the square of the distance of each data value from the mean value.

$$\sigma = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} \tag{18.3}$$

Sample Standard Deviation

Let the sample consist of n elements $\{x_1; x_2; \ldots, x_n\}$, taken from the population, with mean $\bar{x}$. The standard deviation of the sample, denoted by s, is the square root of the average of the squared deviations from the sample mean:

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} \tag{18.4}$$

It is often useful to set your data out in a table so that you can apply the formulae easily. For example to calculate the standard deviation of $\{57; 53; 58; 65; 48; 50; 66; 51\}$, you could set it out in the following way:

$$\begin{aligned} \bar{x} &= \frac{\text{sum of items}}{\text{number of items}} \\ &= \frac{\sum x}{n} \\ &= \frac{448}{8} \\ &= 56 \end{aligned}$$

Note: To get the deviations, subtract each number from the mean.

X	**Deviation** $(X - \bar{X})$	**Deviation squared** $(X - \bar{X})^2$
57	1	1
53	-3	9
58	2	4
65	9	81
48	-8	64
50	-6	36
66	10	100
51	-5	25
$\sum X = 448$	$\sum x = 0$	$\sum (X - \bar{X})^2 = 320$

Note: The sum of the deviations of scores about their mean is zero. This always happens; that is $(X - \bar{X}) = 0$, for any set of data. Why is this? Find out.

Calculate the variance (add the squared results together and divide this total by the number of items).

$$\begin{aligned} \textbf{Variance} &= \frac{\sum (X - \bar{X})^2}{n} \\ &= \frac{320}{8} \\ &= 40 \end{aligned}$$

$$\begin{aligned} \textbf{Standard deviation} &= \sqrt{\text{variance}} \\ &= \sqrt{\frac{\sum (X - \bar{X})^2}{n}} \\ &= \sqrt{\frac{320}{8}} \\ &= \sqrt{40} \\ &= 6.32 \end{aligned}$$

Difference between Population Variance and Sample Variance

As with variance, there is a distinction between the standard deviation, σ, of a whole population and the standard deviation, s, of sample extracted from the population.

When dealing with the complete population the (population) standard deviation is a constant, a parameter which helps to describe the population. When dealing with a sample from the population the (sample) standard deviation varies from sample to sample.

In other words, the standard deviation can be calculated as follows:

1. Calculate the mean value $\bar{x}$.
2. For each data value x_i calculate the difference $x_i - \bar{x}$ between x_i and the mean value $\bar{x}$.
3. Calculate the squares of these differences.
4. Find the average of the squared differences. This quantity is the variance, σ^2.
5. Take the square root of the variance to obtain the standard deviation, σ.

See video: VMfvk at www.everythingmaths.co.za

Example 1: *Variance and Standard Deviation*

QUESTION

What is the variance and standard deviation of the population of possibilities associated with rolling a fair die?

SOLUTION

Step 1 : ***Determine how many outcomes make up the population***
When rolling a fair die, the population consists of 6 possible outcomes. The data set is therefore $x = \{1; 2; 3; 4; 5; 6\}$. and $n = 6$.

Step 2 : ***Calculate the population mean***
The population mean is calculated by:

$$\begin{aligned} \bar{x} &= \frac{1}{6}(1+2+3+4+5+6) \\ &= 3{,}5 \end{aligned}$$

Step 3 : ***Calculate the population variance***
The population variance is calculated by:

$$\begin{aligned} \sigma^2 &= \frac{\sum(x-\bar{x})^2}{n} \\ &= \frac{1}{6}(6{,}25+2{,}25+0{,}25+0{,}25+2{,}25+6{,}25) \\ &= 2{,}917 \end{aligned}$$

Step 4 : ***Alternately the population variance is calculated by:***

X	$(X-\bar{X})$	$(X-\bar{X})^2$
1	-2.5	6.25
2	-1.5	2.25
3	-0.5	0.25
4	0.5	0.25
5	1.5	2.25
6	2.5	6.25
$\sum X = 21$	$\sum x = 0$	$\sum(X-\bar{X})^2 = 17.5$

Step 5 : ***Calculate the standard deviation***
The (population) standard deviation is calculated by:

$$\begin{aligned} \sigma &= \sqrt{2{,}917} \\ &= 1{,}708. \end{aligned}$$

Notice how this standard deviation is somewhere in between the possible deviations.

Interpretation and Application

EMBDV

A large standard deviation indicates that the data values are far from the mean and a small standard deviation indicates that they are clustered closely around the mean.

For example, each of the three samples $(0; 0; 14; 14)$, $(0; 6; 8; 14)$, and $(6; 6; 8; 8)$ has a mean of 7. Their standard deviations are $8{,}08$; $5{,}77$ and $1{,}15$ respectively. The third set has a much smaller standard deviation than the other two because its values are all close to 7. The value of the standard deviation can be considered 'large' or 'small' only in relation to the sample that is being measured. In this case, a standard deviation of 7 may be considered large. Given a different sample, a standard deviation of 7 might be considered small.

Standard deviation may be thought of as a measure of uncertainty. In physical science for example, the reported standard deviation of a group of repeated measurements should give the precision of those measurements. When deciding whether measurements agree with a theoretical prediction, the standard deviation of those measurements is of crucial importance: if the mean of the measurements is too far away from the prediction (with the distance measured in standard deviations), then we consider the measurements as contradicting the prediction. This makes sense since they fall outside the range of values that could reasonably be expected to occur if the prediction were correct and the standard deviation appropriately quantified. (See prediction interval.)

Relationship Between Standard Deviation and the Mean

EMBDW

The mean and the standard deviation of a set of data are usually reported together. In a certain sense, the standard deviation is a "natural" measure of statistical dispersion if the centre of the data is measured about the mean.

Exercise 18 - 1

1. Bridget surveyed the price of petrol at petrol stations in Cape Town and Durban. The raw data, in rands per litre, are given below:

Cape Town	3,96	3,76	4,00	3,91	3,69	3,72
Durban	3,97	3,81	3,52	4,08	3,88	3,68

   (a) Find the mean price in each city and then state which city has the lowest mean.

   (b) Assuming that the data is a population find the standard deviation of each city's prices.

   (c) Assuming the data is a sample find the standard deviation of each city's prices.

   (d) Giving reasons which city has the more consistently priced petrol?

2. The following data represents the pocket money of a sample of teenagers.
   150; 300; 250; 270; 130; 80; 700; 500; 200; 220; 110; 320; 420; 140.
   What is the standard deviation?

3. Consider a set of data that gives the weights of 50 cats at a cat show.

   (a) When is the data seen as a population?

   (b) When is the data seen as a sample?

4. Consider a set of data that gives the results of 20 pupils in a class.

   (a) When is the data seen as a population?

   (b) When is the data seen as a sample?

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(1.) 015e (2.) 015f (3.) 015g (4.) 015h

18.3 Graphical Representation of Measures of Central Tendency and Dispersion

EMBDX

The measures of central tendency (mean, median, mode) and the measures of dispersion (range, semi-inter-quartile range, quartiles, percentiles, inter-quartile range) are numerical methods of summarising data. This section presents methods of representing the summarised data using graphs.

Five Number Summary

EMBDY

One method of summarising a data set is to present a *five number summary*. The five numbers are: minimum, first quartile, median, third quartile and maximum.

Box and Whisker Diagrams

EMBDZ

A *box and whisker* diagram is a method of depicting the five number summary, graphically.

The main features of the box and whisker diagram are shown in Figure 18.2. The box can lie horizontally (as shown) or vertically. For a horizontal diagram, the left edge of the box is placed at the first quartile and the right edge of the box is placed at the third quartile. The height of the box is arbitrary, as there is no y-axis. Inside the box there is some representation of central tendency, with the median shown with a vertical line dividing the box into two. Additionally, a star or asterix is placed at the mean value, centred in the box in the vertical direction. The whiskers which extend to the sides reach the minimum and maximum values.

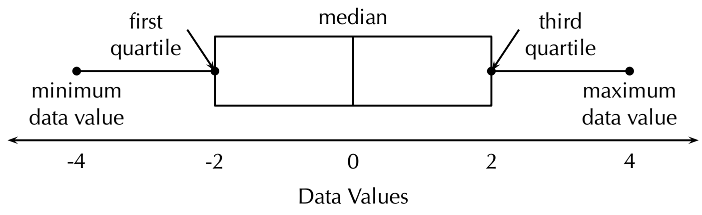


Figure 18.2: Main features of a box and whisker diagram

Example 2: *Box and Whisker Diagram*

QUESTION

Draw a box and whisker diagram for the data set
$x = \{1{,}25;\ 1{,}5;\ 2{,}5;\ 2{,}5;\ 3{,}1;\ 3{,}2;\ 4{,}1;\ 4{,}25;\ 4{,}75;\ 4{,}8;\ 4{,}95;\ 5{,}1\}$.

SOLUTION

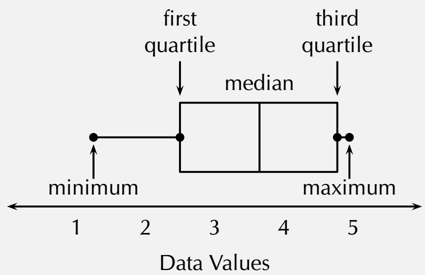
Step 1 : ***Determine the five number summary***

Minimum = $1{,}25$
Maximum = $5{,}10$
Position of first quartile = between 3 and 4
Position of second quartile = between 6 and 7
Position of third quartile = between 9 and 10

Data value between 3 and 4 = $\frac{1}{2}(2{,}5 + 2{,}5) = 2{,}5$
Data value between 6 and 7 = $\frac{1}{2}(3{,}2 + 4{,}1) = 3{,}65$
Data value between 9 and 10 = $\frac{1}{2}(4{,}75 + 4{,}8) = 4{,}775$

The five number summary is therefore: $1{,}25;\ 2{,}5;\ 3{,}65;\ 4{,}775;\ 5{,}10$.

Step 2 : ***Draw a box and whisker diagram and mark the positions of the minimum, maximum and quartiles.***



See video: VMfzi at www.everythingmaths.co.za

Exercise 18 - 2

1. Lisa works as a telesales person. She keeps a record of the number of sales she makes each month. The data below show how much she sells each month.
   $49;\ 12;\ 22;\ 35;\ 2;\ 45;\ 60;\ 48;\ 19;\ 1;\ 43;\ 12$
   Give a five number summary and a box and whisker plot of her sales.

2. Jason is working in a computer store. He sells the following number of computers each month:
   $27;\ 39;\ 3;\ 15;\ 43;\ 27;\ 19;\ 54;\ 65;\ 23;\ 45;\ 16$
   Give a five number summary and a box and whisker plot of his sales,

3. The number of rugby matches attended by 36 season ticket holders is as follows:
   $15;\ 11;\ 7;\ 34;\ 24;\ 22;\ 31;\ 12;\ 9$
   $12;\ 9;\ 1;\ 3;\ 15;\ 5;\ 8;\ 11;\ 2$
   $25;\ 2;\ 6;\ 18;\ 16;\ 17;\ 20;\ 13;\ 17$
   $14;\ 13;\ 11;\ 5;\ 3;\ 2;\ 23;\ 26;\ 40$

   (a) Sum the data.
   (b) Using an appropriate graphical method (give reasons) represent the data.
   (c) Find the median, mode and mean.
   (d) Calculate the five number summary and make a box and whisker plot.
   (e) What is the variance and standard deviation?
   (f) Comment on the data's spread.
   (g) Where are 95% of the results expected to lie?

4. Rose has worked in a florists shop for nine months. She sold the following number of wedding bouquets:
   $16;\ 14;\ 8;\ 12;\ 6;\ 5;\ 3;\ 5;\ 7$

   (a) What is the five-number summary of the data?
   (b) Since there is an odd number of data points what do you observe when calculating the five-numbers?

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(1.) 015i (2.) 015j (3.) 015k (4.) 015m

Cumulative Histograms

EMBEA

Cumulative histograms, also known as ogives, are a plot of cumulative frequency and are used to determine how many data values lie above or below a particular value in a data set. The cumulative frequency is calculated from a frequency table, by adding each frequency to the total of the frequencies of all data values before it in the data set. The last value for the cumulative frequency will always be equal to the total number of data values, since all frequencies will already have been added to the previous total. The cumulative frequency is plotted at the upper limit of the interval.

For example, the cumulative frequencies for Data Set 2 are shown in Table 18.2 and is drawn in Figure 18.3.

Intervals	$0 < n \leq 1$	$1 < n \leq 2$	$2 < n \leq 3$	$3 < n \leq 4$	$4 < n \leq 5$	$5 < n \leq 6$
Frequency	30	32	35	34	37	32
Cumulative Frequency	30	$30 + 32$	$30+32+35$	$30 + 32 + 35 + 34$	$30 + 32 + 35+34+37$	$30 + 32 + 35 + 34 + 37 + 32$
	30	62	97	131	168	200

Table 18.1: Cumulative Frequencies for Data Set 2.

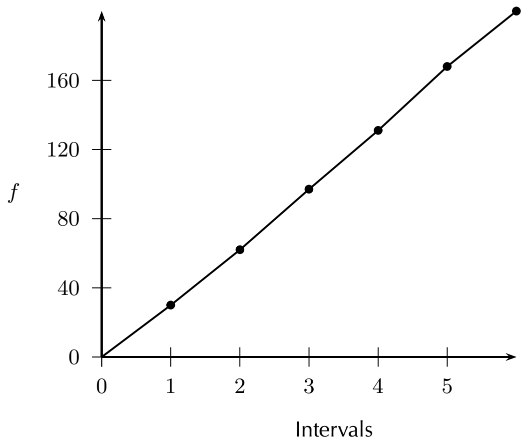

Figure 18.3: Example of a cumulative histogram for Data Set 2.

Notice the frequencies plotted at the upper limit of the intervals, so the points $(30; 1)$ $(62; 2)$ $(97; 3)$, etc have been plotted. This is different from the frequency polygon where we plot frequencies at the midpoints of the intervals.

Exercise 18 - 3

1. Use the following data of peoples ages to answer the questions.
   $2;\ 5;\ 1;\ 76;\ 34;\ 23;\ 65;\ 22;\ 63;\ 45;\ 53;\ 38$
   $4;\ 28;\ 5;\ 73;\ 80;\ 17;\ 15;\ 5;\ 34;\ 37;\ 45;\ 56$
   (a) Using an interval width of 8 construct a cumulative frequency distribution
   (b) How many are below 30?
   (c) How many are below 60?
   (d) Giving an explanation state below what value the bottom 50% of the ages fall
   (e) Below what value do the bottom 40% fall?
   (f) Construct a frequency polygon and an ogive.
   (g) Compare these two plots
2. The weights of bags of sand in grams is given below (rounded to the nearest tenth):
   $50.1;\ 40.4;\ 48.5;\ 29.4;\ 50.2;\ 55.3;\ 58.1;\ 35.3;\ 54.2;\ 43.5$
   $60.1;\ 43.9;\ 45.3;\ 49.2;\ 36.6;\ 31.5;\ 63.1;\ 49.3;\ 43.4;\ 54.1$

(a) Decide on an interval width and state what you observe about your choice.
(b) Give your lowest interval.
(c) Give your highest interval.
(d) Construct a cumulative frequency graph and a frequency polygon.
(e) Compare the cumulative frequency graph and frequency polygon.
(f) Below what value do 53% of the cases fall?
(g) Below what value of 60% of the cases fall?

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(1.) 015n (2.) 015p

18.4 Distribution of Data

EMBEB

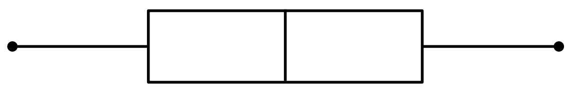
Symmetric and Skewed Data

EMBEC

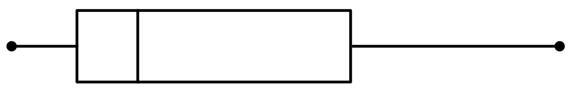
The shape of a data set is important to know.

> *DEFINITION: Shape of a data set*
>
> This describes how the data is distributed relative to the mean and median.

- Symmetrical data sets are balanced on either side of the median.



- Skewed data is spread out on one side more than on the other. It can be skewed right or skewed left.



skewed right

skewed left

Relationship of the Mean, Median, and Mode

EMBED

The relationship of the mean, median, and mode to each other can provide some information about the relative shape of the data distribution. If the mean, median, and mode are approximately equal to each other, the distribution can be assumed to be approximately symmetrical. With both the mean and median known, the following can be concluded:

- (mean - median) ≈ 0 then the data is symmetrical
- (mean - median) > 0 then the data is positively skewed (skewed to the right). This means that the median is close to the start of the data set.
- (mean - median) < 0 then the data is negatively skewed (skewed to the left). This means that the median is close to the end of the data set.

Exercise 18 - 4

1. Three sets of 12 pupils each had test score recorded. The test was out of 50. Use the given data to answer the following questions.

Set A	Set B	Set C
25	32	43
47	34	47
15	35	16
17	32	43
16	25	38
26	16	44
c 24	38	42
27	47	50
22	43	50
24	29	44
12	18	43
31	25	42

Table 18.2: Cumulative Frequencies for Data Set 2.

   (a) For each of the sets calculate the mean and the five number summary.
   (b) For each of the classes find the difference between the mean and the median. Make box and whisker plots on the same set of axes.
   (c) State which of the three are skewed (either right or left).
   (d) Is set A skewed or symmetrical?
   (e) Is set C symmetrical? Why or why not?

2. Two data sets have the same range and interquartile range, but one is skewed right and the other is skewed left. Sketch the box and whisker plots and then invent data (6 points in each set) that meets the requirements.

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(1.) 015q (2.) 015r

18.5 Scatter Plots

EMBEE

A scatter-plot is a graph that shows the relationship between two variables. We say this is bivariate data and we plot the data from two different sets using ordered pairs. For example, we could have mass on the horizontal axis (first variable) and height on the second axis (second variable), or we could have current on the horizontal axis and voltage on the vertical axis.

Ohm's Law is an important relationship in physics. Ohm's law describes the relationship between current and voltage in a conductor, like a piece of wire. When we measure the voltage (dependent variable) that results from a certain current (independent variable) in a wire, we get the data points as shown in Table 18.3.

Table 18.3: Values of current and voltage measured in a wire.

Current	Voltage	Current	Voltage
0	0.4	2.4	1.4
0.2	0.3	2.6	1.6
0.4	0.6	2.8	1.9
0.6	0.6	3	1.9
0.8	0.4	3.2	2
1	1	3.4	1.9
1.2	0.9	3.6	2.1
1.4	0.7	3.8	2.1
1.6	1	4	2.4
1.8	1.1	4.2	2.4
2	1.3	4.4	2.5
2.2	1.1	4.6	2.5

When we plot this data as points, we get the scatter plot shown in Figure 18.4.

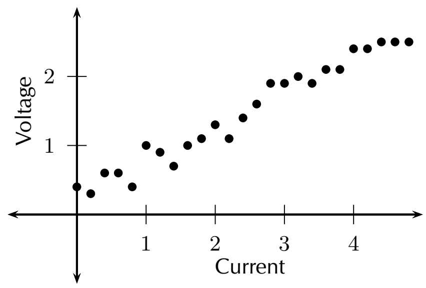

Figure 18.4: Example of a scatter plot

If we are to come up with a function that best describes the data, we would have to say that a straight line best describes this data.

Extension: *Ohm's Law*

Ohm's Law describes the relationship between current and voltage in a conductor. The gradient of the graph of voltage vs. current is known as the *resistance* of the conductor.

Activity: *Scatter Plot*

The function that best describes a set of data can take any form. We will restrict ourselves to the forms already studied, that is, linear, quadratic or exponential. Plot the following sets of data as scatter plots and deduce the type of function that best describes the data. The type of function can either be quadratic or exponential.

1.

x	y	x	y	x	y	x	y
-5	9.8	0	14.2	-2.5	11.9	2.5	49.3
-4.5	4.4	0.5	22.5	-2	6.9	3	68.9
-4	7.6	1	21.5	-1.5	8.2	3.5	88.4
-3.5	7.9	1.5	27.5	-1	7.8	4	117.2
-3	7.5	2	41.9	-0.5	14.4	4.5	151.4

2.

x	y	x	y	x	y	x	y
-5	75	0	5	-2.5	27.5	2.5	7.5
-4.5	63.5	0.5	3.5	-2	21	3	11
-4	53	1	3	-1.5	15.5	3.5	15.5
-3.5	43.5	1.5	3.5	-1	11	4	21
-3	35	2	5	-0.5	7.5	4.5	27.5

3.

Height (cm)	147 168	150 170	152 173	155 175	157 178	160 180	163 183	165
Weight (kg)	52 63	53 64	54 66	56 68	57 70	59 72	60 74	61

DEFINITION: outlier

A point on a scatter plot which is widely separated from the other points or a result differing greatly from others in the same sample is called an outlier.

See video: VMgao at www.everythingmaths.co.za

Exercise 18 - 5

1. A class's results for a test were recorded along with the amount of time spent studying for it. The results are given below.

Score (percent)	Time spent studying (minutes)
67	100
55	85
70	150
90	180
45	70
75	160
50	80
60	90
84	110
30	60
66	96
96	200

(a) Draw a diagram labelling horizontal and vertical axes.
(b) State with reasons, the cause or independent variable and the effect or dependent variable.
(c) Plot the data pairs
(d) What do you observe about the plot?
(e) Is there any pattern emerging?

2. The rankings of eight tennis players is given along with the time they spend practising.

Practise time (min)	Ranking
154	5
390	1
130	6
70	8
240	3
280	2
175	4
103	7

(a) Construct a scatter plot and explain how you chose the dependent (cause) and independent (effect) variables.
(b) What pattern or trend do you observe?

3. Eight children's sweet consumption and sleep habits were recorded. The data is given in the following table.

Number of sweets (per week)	Average sleeping time (per day)
15	4
12	4.5
5	8
3	8.5
18	3
23	2
11	5
4	8

(a) What is the dependent (cause) variable?
(b) What is the independent (effect) variable?
(c) Construct a scatter plot of the data.
(d) What trend do you observe?

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(1.) 015s (2.) 015t (3.) 015u

18.6 Misuse of Statistics

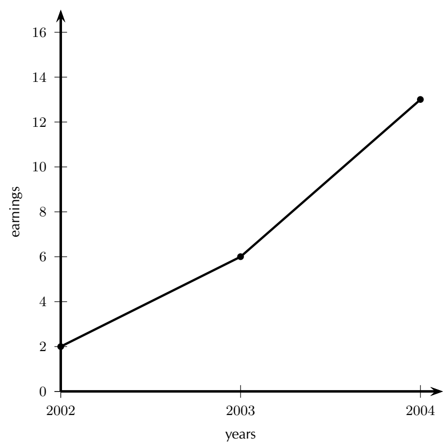
EMBEF

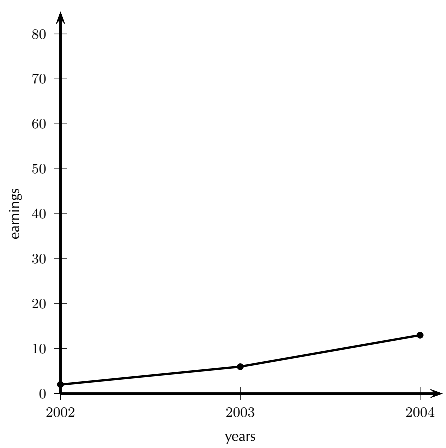
Statistics can be manipulated in many ways that can be misleading. Graphs need to be carefully analysed and questions must always be asked about "the story behind the figures." Potential manipulations are:

1. Changing the scale to change the appearance of a graph
2. Omissions and biased selection of data
3. Focus on particular research questions
4. Selection of groups

Activity: *Misuse of statistics*

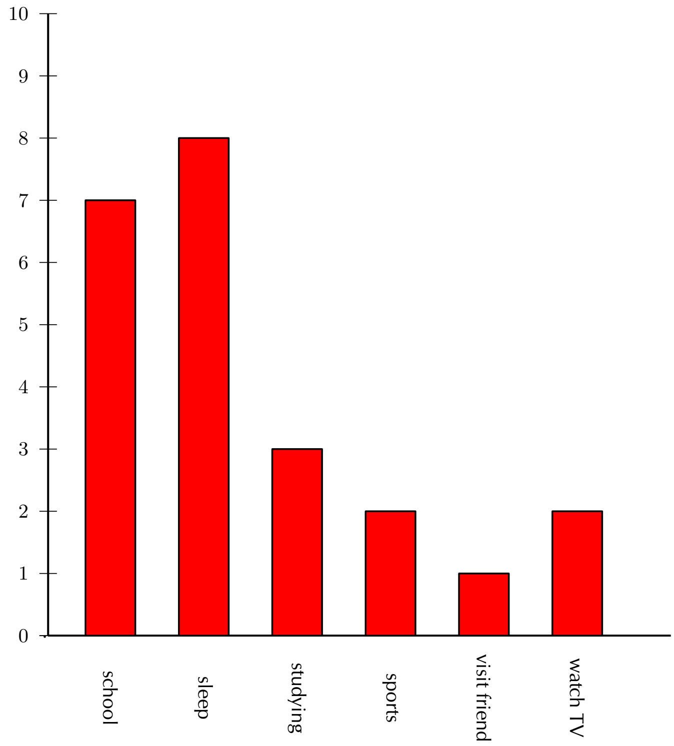
1. Examine the following graphs and comment on the effects of changing scale.





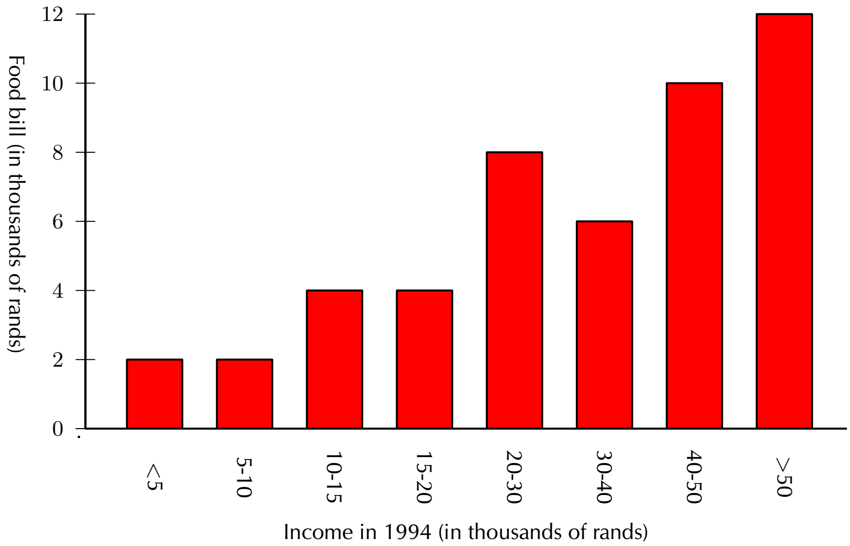
2. Examine the following three plots and comment on omission, selection and bias. Hint: What is wrong with the data and what is missing from the bar and pie charts?

Activity	Hours
Sleep	8
Sports	2
School	7
Visit friend	1
Watch TV	2
Studying	3



Exercise 18 - 6

The bar graph below shows the results of a study that looked at the cost of food compared to the income of a household in 1994.



Income (thousands of rands)	Food bill (thousands of rands)
< 5	2
$5 - 10$	2
$10 - 15$	4
$15 - 20$	4
$20 - 30$	8
$30 - 40$	6
$40 - 50$	10
> 50	12

1. What is the dependent variable? Why?
2. What conclusion can you make about this variable? Why? Does this make sense?
3. What would happen if the graph was changed from food bill in thousands of rands to percentage of income?
4. Construct this bar graph using a table. What conclusions can be drawn?
5. Why do the two graphs differ despite showing the same information?
6. What else is observed? Does this affect the fairness of the results?

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(1.) 015v

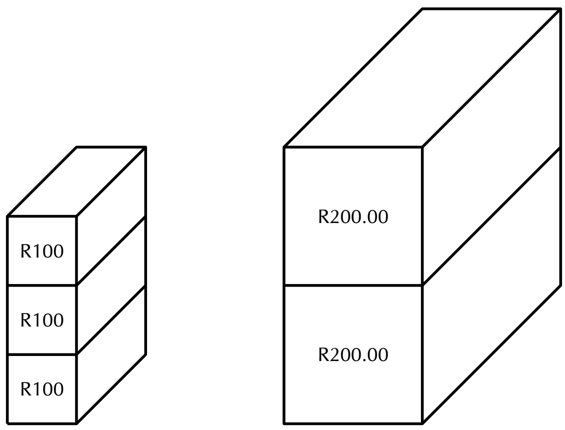
Chapter 18 End of Chapter Exercises

1. Many accidents occur during the holidays between Durban and Johannesburg. A study was done to see if speeding was a factor in the high accident rate. Use the results given to answer the following questions.

Speed (km/h)	Frequency
$60 < x \leq 70$	3
$70 < x \leq 80$	2
$80 < x \leq 90$	6
$90 < x \leq 100$	40
$100 < x \leq 110$	50
$110 < x \leq 120$	30
$120 < x \leq 130$	15
$130 < x \leq 140$	12
$140 < x \leq 150$	3
$150 < x \leq 160$	2

   (a) Draw a graph to illustrate this information.
   (b) Use your graph to find the median speed and the interquartile range.
   (c) What percent of cars travel more than 120 km/h on this road?
   (d) Do cars generally exceed the speed limit?

2. The following two diagrams (showing two schools contribution to charity) have been exaggerated. Explain how they are misleading and redraw them so that they are not misleading.



3. The monthly income of eight teachers are given as follows:
R10 050; R14 300; R9 800; R15 000; R12 140; R13 800; R11 990; R12 900.

(a) What is the mean income and the standard deviation?
(b) How many of the salaries are within one standard deviation of the mean?
(c) If each teacher gets a bonus of R500 added to their pay what is the new mean and standard deviation?
(d) If each teacher gets a bonus of 10% on their salary what is the new mean and standard deviation?
(e) Determine for both of the above, how many salaries are within one standard deviation of the mean.
(f) Using the above information work out which bonus is more beneficial financially for the teachers.

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(1.) 0161 (2.) 0162 (3.) 0163

Independent and Dependent Events 19

19.1 Introduction

EMBEG

In probability theory events are either independent or dependent. This chapter discusses the differences between these two categories of events and will show that we use different sets of mathematical rules for handling them.

See introductory video: VMgdw at www.everythingmaths.co.za

19.2 Definitions

EMBEH

Two events are independent if knowing something about the value of one event does not give any information about the value of the second event. For example, the event of getting a "1" when a die is rolled and the event of getting a "1" the second time it is thrown are independent.

The probability of two independent events occurring, $P(A \cap B)$, is given by:

$$P(A \cap B) = P(A) \times P(B) \quad (19.1)$$

DEFINITION: Independent events

Events are said to be independent if the result or outcome of one event does not affect the result or outcome of the other event. So $P(A/C) = P(A)$, where $P(A/C)$ represents the probability of event A after event C has occurred.

Example 1: *Independent Events*

QUESTION

What is the probability of rolling a 1 and then rolling a 6 on a fair die?

SOLUTION

Step 1 : ***Identify the two events and determine whether the events are independent or not***

Event A is rolling a 1 and event B is rolling a 6. Since the outcome of the first event does not affect the outcome of the second event, the events are independent.

Step 2 : ***Determine the probability of the specific outcomes occurring, for each event***

The probability of rolling a 1 is $\frac{1}{6}$ and the probability of rolling a 6 is $\frac{1}{6}$. Therefore, $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{6}$.

Step 3 : ***Use equation 19.1 to determine the probability of the two events occurring together.***

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ &= \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{36} \end{aligned}$$

The probability of rolling a 1 and then rolling a 6 on a fair die is $\frac{1}{36}$.

Consequently, two events are dependent if the outcome of the first event affects the outcome of the second event.

DEFINITION: *Dependent events*

Two events are dependent if the outcome of one event is affected by the outcome of the other event i.e. $P(A/C) \neq P(A)$.

Example 2: *Dependent Events*

QUESTION

A cloth bag has four coins, one R1 coin, two R2 coins and one R5 coin. What is the probability of first selecting a R1 coin and then selecting a R2 coin?

SOLUTION

Step 1 : ***Identify the two events and determine whether the events are independent or not***

Event A is selecting a R1 coin and event B is next selecting a R2. Since the outcome of the first event affects the outcome of the second event (because there are less coins to choose from after the first coin has been selected), the events are dependent.

Step 2 : ***Determine the probability of the specific outcomes occurring, for each event***

The probability of first selecting a R1 coin is $\frac{1}{4}$ and the probability of next selecting a R2 coin is $\frac{2}{3}$ (because after the R1 coin has been selected, there are only three coins to choose from).

Therefore, $P(A) = \frac{1}{4}$ and $P(B) = \frac{2}{3}$.

Step 3 : ***Use equation 19.1 to determine the probability of the two events occurring together.***

The same equation as for independent events are used, but the probabilities are calculated differently.

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ &= \frac{1}{4} \times \frac{2}{3} \\ &= \frac{2}{12} \\ &= \frac{1}{6} \end{aligned}$$

The probability of first selecting a R1 coin followed by selecting a R2 coin is $\frac{1}{6}$.

Identification of Independent and Dependent Events EMBEI

Use of a Contingency Table

A two-way contingency table (studied in an earlier grade) can be used to determine whether events are independent or dependent.

DEFINITION: two-way contingency table

A two-way contingency table is used to represent possible outcomes when two events are combined in a statistical analysis.

For example we can draw and analyse a two-way contingency table to solve the following problem.

Example 3: *Contingency Tables*

QUESTION

A medical trial into the effectiveness of a new medication was carried out. 120 males and 90 females responded. Out of these 50 males and 40 females responded positively to the medication.

1. *Was the medication's success independent of gender? Explain.*
2. *Give a table for the independence of gender results.*

SOLUTION

Step 1 : ***Draw a contingency table***

	Male	Female	Totals
Positive result	50	40	90
No Positive result	70	50	120
Totals	120	90	210

Step 2 : ***Work out probabilities***

$$\text{P(male).P(positive result)} = \frac{120}{210} = 0{,}57$$
$$\text{P(female).P(positive result)} = \frac{90}{210} = 0{,}43$$
$$\text{P(male and positive result)} = \frac{50}{210} = 0{,}24$$

Step 3 : ***Draw conclusion***

P(male and positive result) is the observed probability and P(male).P(positive result) is the expected probability. These two are quite different. So there is no evidence that the medication's success is independent of gender.

Step 4 : ***Gender-independent results***

To get gender independence we need the positive results in the same ratio as the gender. The gender ratio is: $120 : 90$, or $4 : 3$, so the number in the male and positive column would have to be $\frac{4}{7}$ of the total number of patients responding positively which gives $51{,}4$. This leads to the following table:

	Male	Female	Totals
Positive result	51,4	38,6	90
No Positive result	68,6	51,4	120
Totals	120	90	210

Use of a Venn Diagram

We can also use Venn diagrams to check whether events are dependent or independent.

Also note that $P(A/C) = \frac{P(A \cap C)}{P(C)}$. For example, we can draw a Venn diagram and a contingency table to illustrate and analyse the following example.

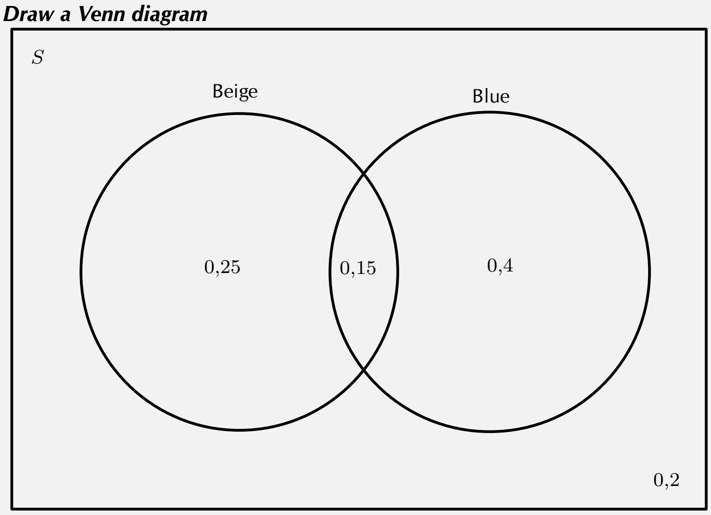
Example 4: *Venn diagrams and events*

QUESTION

A school decided that its uniform needed upgrading. The colours on offer were beige or blue or beige and blue. 40% of the school wanted beige, 55% wanted blue and 15%said a combination would be fine. Are the two events independent?

SOLUTION

Step 1 : ***Draw a Venn diagram***



Step 2 : ***Draw up a contingency table***

	Beige	Not Beige	Totals
Blue	0,15	0,4	0,55
Not Blue	0,25	0,2	0,35
Totals	0,40	0,6	1

Step 3 : ***Work out the probabilities***

P(Blue)$= 0{,}4$; P(Beige)$= 0{,}55$; P(Both)$= 0{,}15$; P(Neither)$= 0{,}20$

Probability of choosing beige after blue is:

$$
\begin{aligned}
P\left(\frac{\text{Beige}}{\text{Blue}}\right) &= \frac{P(\text{Beige} \cap \text{Blue})}{P(\text{Blue})} \\
&= \frac{0{,}15}{0{,}55} \\
&= 0{,}27
\end{aligned}
$$

Step 4 : ***Solve the problem***

Since $P\left(\frac{\text{Beige}}{\text{Blue}}\right)$ the events are statistically dependent.

Extension:	*Applications of Probability Theory*

Two major applications of probability theory in everyday life are in risk assessment and in trade on commodity markets. Governments typically apply probability methods in environmental regulation where it is called "pathway analysis", and are often measuring well-being using methods that are stochastic in nature, and choosing projects to undertake based on statistical analyses of their probable effect on the population as a whole. It is not correct to say that statistics are involved in the modelling itself, as typically the assessments of risk are one-time and thus require more fundamental probability models, e.g. "the probability of another 9/11". A law of small numbers tends to apply to all such choices and perception of the effect of such choices, which makes probability measures a political matter.

A good example is the effect of the perceived probability of any widespread Middle East conflict on oil prices - which have ripple effects in the economy as a whole. An assessment by a commodity trade that a war is more likely vs. less likely sends prices up or down, and signals other traders of that opinion. Accordingly, the probabilities are not assessed independently nor necessarily very rationally. The theory of behavioural finance emerged to describe the effect of such groupthink on pricing, on policy, and on peace and conflict.

It can reasonably be said that the discovery of rigorous methods to assess and combine probability assessments has had a profound effect on modern society. A good example is the application of game theory, itself based strictly on probability, to the Cold War and the mutual assured destruction doctrine. Accordingly, it may be of some importance to most citizens to understand how odds and probability assessments are made, and how they contribute to reputations and to decisions, especially in a democracy.

Another significant application of probability theory in everyday life is reliability. Many consumer products, such as automobiles and consumer electronics, utilise reliability theory in the design of the product in order to reduce the probability of failure. The probability of failure is also closely associated with the product's warranty.

Chapter 19 End of Chapter Exercises

1. In each of the following contingency tables give the expected numbers for the events to be perfectly independent and decide if the events are independent or dependent.

(a)

	Brown eyes	Not Brown eyes	Totals
Black hair	50	30	80
Red hair	70	80	150
Totals	120	110	230

(b)

	Point A	Point B	Totals
Busses left late	15	40	55
Buses left on time	25	20	45
Totals	40	60	100

(c)

	Durban	Bloemfontein	Totals
Liked living there	130	30	160
Did not like living there	140	200	340
Totals	270	230	500

(d)

	Multivitamin A	Multivitamin B	Totals
Improvement in health	400	300	700
No improvement in health	140	120	260
Totals	540	420	960

2. A study was undertaken to see how many people in Port Elizabeth owned either a Volkswagen or a Toyota. 3% owned both, 25% owned a Toyota and 60% owned a Volkswagen. Draw a contingency table to show all events and decide if car ownership is independent.
3. Jane invested in the stock market. The probability that she will not lose all her money is 0,32. What is the probability that she will lose all her money? Explain.
4. If D and F are mutually exclusive events, with $P(D') = 0{,}3$ and $P(D \text{ or } F) = 0{,}94$, find $P(F)$.
5. A car sales person has pink, lime-green and purple models of car A and purple, orange and multicolour models of car B. One dark night a thief steals a car.
   (a) What is the experiment and sample space?
   (b) Draw a Venn diagram to show this.
   (c) What is the probability of stealing either a model of A or a model of B?
   (d) What is the probability of stealing both a model of A and a model of B?
6. The probability of Event X is $0{,}43$ and the probability of Event Y is $0{,}24$. The probability of both occurring together is $0{,}10$. What is the probability that X or Y will occur (this includes X and Y occurring simultaneously)?
7. $P(H) = 0{,}62$; $P(J) = 0{,}39$ and$P(H \text{ and } J) = 0{,}31$. Calculate:
   (a) $P(H')$
   (b) $P(H \text{ or } J)$
   (c) $P(H' \text{ or } J')$
   (d) $P(H' \text{ or } J)$
   (e) $P(H' \text{ and } J')$
8. The last ten letters of the alphabet were placed in a hat and people were asked to pick one of them. Event D is picking a vowel, Event E is picking a consonant and Event F is picking the last four letters. Calculate the following probabilities:
   (a) $P(F')$
   (b) $P(F \text{ or } D)$
   (c) $P(\text{neither } E \text{ nor } F)$
   (d) $P(D \text{ and } E)$
   (e) $P(E \text{ and } F)$
   (f) $P(E \text{ and } D')$
9. At Dawnview High there are 400 Grade 12's. 270 do Computer Science, 300 do English and 50 do Typing. All those doing Computer Science do English, 20 take Computer Science and Typing and 35 take English and Typing. Using a Venn diagram calculate the probability that a pupil drawn at random will take:
   (a) English, but not Typing or Computer Science

(b) English but not Typing
(c) English and Typing but not Computer Science
(d) English or Typing

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(1.) 0164 (2.) 0165 (3.) 0166 (4.) 0167 (5.) 0168 (6.) 0169
(7.) 016a (8.) 016b (9.) 016c

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